19

PROBABILITY

311en19

In our daily life, we often used phrases such as 'It may rain today', or 'India may win the match' or ' I may be selected for this post.' These phrases involve an element of uncertainty. How can we measure this uncertainty? A measure of this uncertainty is provided by a branch of Mathematics, called the theory of probability. Probability Theory is designed to measure the degree of uncertainty regarding the happening of a given event. The dictionary meaning of probability is ' likely though not certain to occur. Thus, when a coin is tossed, a head is likely to occur but may not occur. Similarly, when a die is thrown, it may or may not show the number 6.

In this lesson we shall discuss some basic concepts of probability, addition theorem, dependent and independent events, multiplication theorem, Baye's theorem, ramdom variable, its probability distribution and binomial distribution.

OBJECTIVES

After studying this lesson, you will be able to :

- define probability of occurance of an event;
- cite through examples that probability of occurance of an event is a non-negative fraction, not greater than one;
- use permutation and combinations in solving problems in probability;
- state and establish the addition theorems on probability and the conditions under which each holds;
- generalize the addition theorem of probability for mutually exclusive events;
- understand multiplication law for independent and dependent events and solve problems releated to them.
- understand conditional probability and solve problems releated to it.
- understand Baye's theorem and solve questions related to it.
- define random variable and find its probability distribution.
- understand and find, mean and variance of random variable.
- understand binomial distribution and solve questions based on it.

EXPECTED BACKGROUND KNOWLEDGE

- Knowledge of random experiments and events.
- The meaning of sample space.

- A standard deck of playing cards consists of 52 cards divided into 4 suits of 13 cards each : spades, hearts, diamonds, clubs and cards in each suit are - ace, king, queen, jack, 10, 9, 8, 7, 6, 5, 4, 3 and 2. Kings, Queens and Jacks are called face cards and the other cards are called number cards.

19.1 EVENTS AND THEIR PROBABILITY

In the previous lesson, we have learnt whether an activity is a random experiment or not. The study of probability always refers to random experiments. Hence, from now onwards, the word experiment will be used for a random experiment only. In the preceeding lesson, we have defined different types of events such as equally likely, mutually exclusive, exhaustive, independent and dependent events and cited examples of the above mentioned events.

Here we are interested in the chance that a particular event will occur, when an experiment is performed. Let us consider some examples.

What are the chances of getting a ' Head' in tossing an unbiased coin ? There are only two equally likely outcomes, namely head and tail. In our day to day language, we say that the coin has chance 1 in 2 of showing up a head. In technical language, we say that the probability of getting a head is $\frac{1}{2}$.

Similarly, in the experiment of rolling a die, there are six equally likely outcomes 1, 2,3,4,5 or 6. The face with number '1' (say) has chance 1 in 6 of appearing on the top. Thus, we say that the probability of getting 1 is $\frac{1}{6}$.

In the above experiment, suppose we are interested in finding the probability of getting even number on the top, when a die is rolled. Clearly, the possible numbers are 2, 4 and 6 and the chance of getting an even number is 3 in 6. Thus, we say that the probability of getting an even number is $\frac{3}{6}$, i.e., $\frac{1}{2}$.

The above discussion suggests the following definition of probability.

If an experiment with '*n*' exhaustive, mutually exclusive and equally likely outcomes, *m* outcomes are favourable to the happening of an event A, the probability '*p*' of happening of A is given by

$$p = \mathrm{P}(\mathrm{A}) = \frac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}} = \frac{m}{n} \qquad \text{....(i)}$$

Since the number of cases favourable to the non-happening of the event A are $n - m$, the probability 'q' that 'A' will not happen is given by

$$q = \frac{n-m}{n} = 1 - \frac{m}{n}$$

$$= 1 - p \quad \text{[Using (i)]}$$

$$\therefore \qquad p + q = 1.$$

Obviously, p as well as q are non-negative and cannot exceed unity.

i.e., $0 \le p \le 1, \quad 0 \le q \le 1$

Thus, the probability of occurrence of an event lies between 0 and 1[including 0 and 1].

Remarks

1. Probability 'p' of the happening of an event is known as the probability of success and the probability 'q' of the non-happening of the event as the probability of failure.
2. Probability of an impossible event is 0 and that of a sure event is 1

   if P (A) = 1, the event A is certainly going to happen and

   if P (A) = 0, the event is certainly not going to happen.
3. The number (m) of favourable outcomes to an event cannot be greater than the total number of outcomes (n).

Let us consider some examples

Example 19.1 In a simultaneous toss of two coins, find the probability of

(i) getting 2 heads (ii) exactly 1 head

Solution : Here, the possible outcomes are

HH, HT, TH, TT.

i.e., Total number of possible outcomes = 4.

(i) Number of outcomes favourable to the event (2 heads) = 1 (i.e., HH).

$$\therefore \qquad P\,(\text{ 2 heads }) = \frac{1}{4}.$$

(ii) Now the event consisting of exactly one head has two favourable cases,

namely HT and TH. $\quad \therefore \quad P\,(\text{ exactly one head }) = \frac{2}{4} = \frac{1}{2}.$

Example 19.2 In a single throw of two dice, what is the probability that the sum is 9?

Solution : The number of possible outcomes is $6 \times 6 = 36$. We write them as given below :

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

Now, how do we get a total of 9. We have :

$$3 + 6 = 9,\ 4 + 5 = 9,\ 5 + 4 = 9,\ 6 + 3 = 9$$

In other words, the outcomes (3, 6), (4, 5), (5, 4) and (6, 3) are favourable to the said event, i.e., the number of favourable outcomes is 4.

Hence, P (a total of 9) = $\frac{4}{36} = \frac{1}{9}$

Example 19.3 What is the chance that a leap year, selected at random, will contain 53 Sundays?

Solution : A leap year consists of 366 days consisting of 52 weeks and 2 extra days. These two extra days can occur in the following possible ways.

(i) Sunday and Monday
(ii) Monday and Tuesday
(iii) Tuesday and Wednesday
(iv) Wednesday and Thursday
(v) Thursday and Friday
(vi) Friday and Saturday
(vii) Saturday and Sunday

Out of the above seven possibilities, two outcomes,

e.g., (i) and (vii), are favourable to the event

$\therefore$ P (53 Sundays) = $\frac{2}{7}$

CHECK YOUR PROGRESS 19.1

1. A die is rolled once. Find the probability of getting 3.
2. A coin is tossed once. What is the probability of getting the tail ?
3. What is the probability of the die coming up with a number greater than 3 ?
4. In a simultaneous toss of two coins, find the probability of getting ' at least' one tail.
5. From a bag containing 15 red and 10 blue balls, a ball is drawn 'at random'. What is the probability of drawing (i) a red ball ? (ii) a blue ball ?
6. If two dice are thrown, what is the probability that the sum is (i) 6 ? (ii) 8? (iii) 10? (iv) 12?
7. If two dice are thrown, what is the probability that the sum of the numbers on the two faces is divisible by 3 or by 4 ?
8. If two dice are thrown, what is the probability that the sum of the numbers on the two faces is greater than 10 ?
9. What is the probability of getting a red card from a well shuffled deck of 52 cards ?
10. If a card is selected from a well shuffled deck of 52 cards, what is the probability of drawing

    (i) a spade ? (ii) a king ? (iii) a king of spade ?
11. A pair of dice is thrown. Find the probability of getting

(i) a sum as a prime number (ii) a doublet, i.e., the same number on both dice

(iii) a multiple of 2 on one die and a multiple of 3 on the other.

12. Three coins are tossed simultaneously. Find the probability of getting

(i) no head (ii) at least one head (iii) all heads

19.2. CALCULATION OF PROBABILITY USING COMBINATORICS (PERMUTATIONS AND COMBINATIONS)

In the preceding section, we calculated the probability of an event by listing down all the possible outcomes and the outcomes favourable to the event. This is possible when the number of outcomes is small, otherwise it becomes difficult and time consuming process. In general, we do not require the actual listing of the outcomes, but require only the total number of possible outcomes and the number of outcomes favourable to the event. In many cases, these can be found by applying the knowledge of permutations and combinations, which you have already studied.

Let us consider the following examples :

Example 19.4 A bag contains 3 red, 6 white and 7 blue balls. What is the probability that two balls drawn are white and blue ?

Solution : Total number of balls = 3 + 6 + 7 = 16

Now, out of 16 balls, 2 can be drawn in ${}^{16}C_2$ ways.

$\therefore$ Exhaustive number of cases = ${}^{16}C_2 = \dfrac{16 \times 15}{2} = 120$

Out of 6 white balls, 1 ball can be drawn in ${}^{6}C_1$ ways and out of 7 blue balls, one can be drawn is ${}^{7}C_1$ ways. Since each of the former case is associated with each of the later case, therefore total number of favourable cases are ${}^{6}C_1 \times {}^{7}C_1 = 6 \times 7 = 42$.

$\therefore$ Required probability $= \dfrac{42}{120} = \dfrac{7}{20}$

Remarks

When two or more balls are drawn from a bag containing several balls, there are two ways in which these balls can be drawn.

(i) Without replacement : The ball first drawn is not put back in the bag, when the second ball is drawn. The third ball is also drawn without putting back the balls drawn earlier and so on. Obviously, the case of drawing the balls without replacement is the same as drawing them together.

(ii) With replacement : In this case, the ball drawn is put back in the bag before drawing the next ball. Here the number of balls in the bag remains the same, every time a ball is drawn.

In these types of problems, unless stated otherwise, we consider the problem of without replacement.

Example 19.5 Six cards are drawn at random from a pack of 52 cards. What is the probability that 3 will be red and 3 black?

Solution : Six cards can be drawn from the pack of 52 cards in ${}^{52}C_6$ ways.

i.e., Total number of possible outcomes = ${}^{52}C_6$

3 red cards can be drawn in ${}^{26}C_3$ ways and

3 black cards can be drawn in ${}^{26}C_3$ ways.

$\therefore$ Total number of favourable cases = ${}^{26}C_3 \times {}^{26}C_3$

Hence, the required probability $= \dfrac{{}^{26}C_3 \times {}^{26}C_3}{{}^{52}C_6} = \dfrac{13000}{39151}$

Example 19.6 Four persons are chosen at random from a group of 3 men, 2 women and 4 children. Show that the chance that exactly two of them will be children is $\dfrac{10}{21}$.

Solution : Total number of persons in the group = 3 + 2 + 4 = 9. Four persons are chosen at random. If two of the chosen persons are children, then the remaining two can be chosen from 5 persons (3 men + 2 women).

Number of ways in which 2 children can be selected from 4, children = ${}^{4}C_2 = \dfrac{4\times3}{1\times2} = 6$

Number of ways in which remaining of the two persons can be selected

from 5 persons = ${}^{5}C_2 = \dfrac{5\times4}{1\times2} = 10$

Total number of ways in which 4 persons can be selected out of

9 persons = ${}^{9}C_4 = \dfrac{9\times8\times7\times6}{1\times2\times3\times4} = 126$

Hence, the required probability $= \dfrac{{}^{4}C_2 \times {}^{5}C_2}{{}^{9}C_4} = \dfrac{6\times10}{126} = \dfrac{10}{21}$

CHECK YOUR PROGRESS 19.2

1. A bag contains 3 red, 6 white and 7 blue balls. What is the probability that two balls drawn at random are both white?
2. A bag contains 5 red and 8 blue balls. What is the probability that two balls drawn are red and blue ?
3. A bag contains 20 white and 30 black balls. Find the probability of getting 2 white balls, when two balls are drawn at random

   (a) with replacement (b) without replacement

4. Three cards are drawn from a well-shuffled pack of 52 cards. Find the probability that all the three cards are jacks.

5. Two cards are drawn from a well-shuffled pack of 52 cards. Show that the chances of drawing both aces is $\frac{1}{221}$.

6. In a group of 10 outstanding students in a school, there are 6 boys and 4 girls. Three students are to be selected out of these at random for a debate competition. Find the probability that

   (i) one is boy and two are girls. (ii) all are boys. (iii) all are girls.

7. Out of 21 tickets marked with numbers from 1 to 21, three are drawn at random. Find the probability that the numbers on them are in A.P.

8. Two cards are drawn at random from 8 cards numbered 1 to 8. What is the probability that the sum of the numbers is odd, if the cards are drawn together ?

9. A team of 5 players is to be selected from a group of 6 boys and 8 girls. If the selection is made randomly, find the probability that there are 2 boys and 3 girls in the team.

10. An integer is chosen at random from the first 200 positive integers.Find the probability that the integer is divisible by 6 or 8.

19.3 EVENT RELATIONS

19.3.1 Complement of an event

Let us consider the example of throwing a fair die. The sample space of this experiment is

$$S = \{ 1 , 2, 3, 4, 5, 6 \}$$

If A be the event of getting an even number, then the sample points 2, 4 and 6 are favourable to the event A.

The remaining sample points 1, 3 and 5 are not favourable to the event A. Therefore, these will occur when the event A will not occur.

In an experiment, the outcomes which are not favourable to the event A are called complement of A and defined as follows :

'The outcomes favourable to the complement of an event A consists of all those outcomes which are not favourable to the event A, and are denoted by 'not' A or by $\bar{A}$.

19.3.2 Event 'A or B'

Let us consider the example of throwing a die. A is an event of getting a multiple of 2 and B be another event of getting a multiple of 3.

The outcomes 2, 4 and 6 are favourable to the event A and the outcomes 3 and 6 are favourable to the event B.

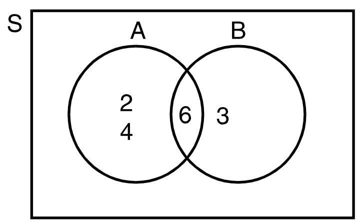


Fig. 19.1

The happening of event A or B is $A \cup B = \{ 2, 3, 4, 6 \}$

Again, if A be the event of getting an even number and B is another event of getting an odd number, then $A = \{ 2, 4, 6 \}$, $B = \{ 1, 3, 5 \}$

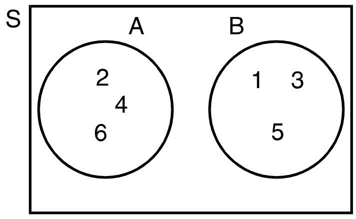


Fig. 19.2

$A \cup B = \{1, 2, 3, 4, 5, 6 \}$

Here, it may be observed that if A and B are two events, then the event 'A or B' ($A \cup B$) will consist of the outcomes which are either favourable to the event A or to the event B or to both the events.

Thus, the event 'A or B' occurs, if either A or B or both occur.

19.3.3 Event "A and B'

Recall the example of throwing a die in which A is the event of getting a multiple of 2 and B is the event of getting a multiple of 3. The outcomes favourable to A are 2, 4, 6 and the outcomes favourable to B are 3 , 6.

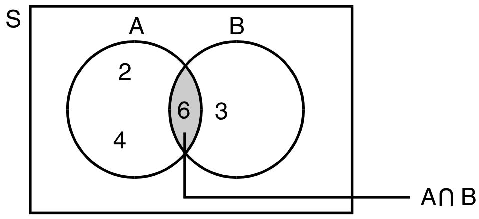


Fig. 19.3

Here, we observe that the outcome 6 is favourable to both the events A and B.

Draw a card from a well shuffled deck of 52 cards. A and B are two events defined as

A : a red card,   B : a king

We know that there are 26 red cards and 4 kings in a deck of cards. Out of these 4 kings, two are red.

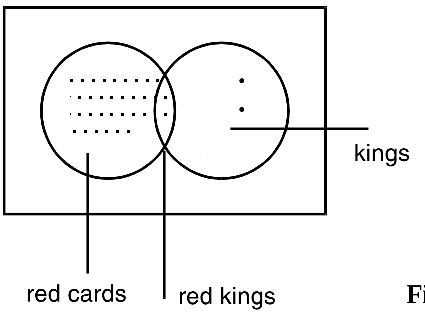


Fig. 19.4

Here, we see that the two red kings are favourable to both the events.

Hence, the event ' A and B' consists of all those outcomes which are favourable to both the events A and B. That is, the event 'A and B' occurs, when both the events A and B occur simultaneously. Symbolically, it is denoted as $A \cap B$.

19.4 ADDITIVE LAW OF PROBABILITY

Let A be the event of getting an odd number and B be the event of getting a prime number in a single throw of a die. What will be the probability that it is either an odd number or a prime number ?

In a single throw of a die, the sample space would be

$$S = \{ 1, 2, 3, 4, 5, 6 \}$$

The outcomes favourable to the events A and B are

$$A = \{ 1, 3, 5 \}, B = \{ 2, 3, 5 \}$$

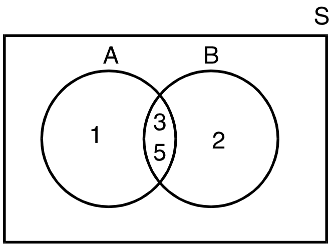


Fig. 19.5

The outcomes favourable to the event 'A or B' are

$$A \cup B = \{1, 2, 3, 5 \}.$$

Thus, the probability of getting either an odd number or a prime number will be

$$P(A \text{ or } B) = \frac{4}{6} = \frac{2}{3}$$

To discover an alternate method, we can proceed as follows :

The outcomes favourable to the event A are 1, 3 and 5. $\therefore$ $P(A) = \frac{3}{6}$

Similarly, $P(B) = \frac{3}{6}$

The outcomes favourable to the event 'A and B' are 3 and 5. $\therefore$ $P(A \text{ and } B) = \frac{2}{6}$

Now, $$P(A) + P(B) - P(A \text{ and } B) = \frac{3}{6} + \frac{3}{6} - \frac{2}{6}$$

$$\frac{4}{6} = \frac{2}{3} = P(A \text{ or } B)$$

Thus, we state the following law, called additive rule, which provides a technique for finding the probability of the union of two events, when they are not disjoint.

For any two events A and B of a sample space S,

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

or $$P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad \text{.....(ii)}$$

Example 19.7 A card is drawn from a well-shuffled deck of 52 cards. What is the prob ability that it is either a spade or a king ?

Solution : If a card is drawn at random from a well-shuffled deck of cards, the likelyhood of any of the 52 cards being drawn is the same. Obviously, the sample space consists of 52 sample points.

If A and B denote the events of drawing a 'spade card' and a 'king' respectively, then the event A consists of 13 sample points, whereas the event B consists of 4 sample points. Therefore,

$$P(A) = \frac{13}{52}, \qquad P(B) = \frac{4}{52}$$

The compound event $(A \cap B)$ consists of only one sample point, viz.; king of spade. So,

$$P(A \cap B) = \frac{1}{52}$$

Hence, the probability that the card drawn is either a spade or a king is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Example 19.8 In an experiment with throwing 2 fair dice, consider the events

A : The sum of numbers on the faces is 8

B : Doubles are thrown.

What is the probability of getting A or B ?

Solution : In a throw of two dice, the sample space consists of $6 \times 6 = 36$ sample points.

The favourable outcomes to the event A (the sum of the numbers on the faces is 8) are

$$A = \{(2,6), (3,5), (4,4), (5,3), (6,2)\}$$

The favourable outcomes to the event B (Double means both dice have the same number) are

$$B = \{(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)\}$$

$$\therefore \quad A \cap B = \{(4,4)\}.$$

Now $P(A) = \frac{5}{36}$, $P(B) = \frac{6}{36}$, $P(A \cap B) = \frac{1}{36}$

Thus, the probability of A or B is

$$P(A \cup B) = \frac{5}{36} + \frac{6}{36} - \frac{1}{36} = \frac{10}{36} = \frac{5}{18}$$

19.5 ADDITIVE LAW OF PROBABILITY FOR MUTUALLY EXCLUSIVE EVENTS

We know that the events A and B are mutually exclusive, if and only if they have no outcomes in common. That is, for mutually exclusive events,

$$P(A \text{ and } B) = 0$$

Substituting this value in the additive law of probability, we get the following law :

$$P(A \text{ or } B) = P(A) + P(B) \qquad \text{.....(iii)}$$

Example 19.9 In a single throw of two dice, find the probability of a total of 9 or 11.

Solution : Clearly, the events - a total of 9 and a total of 11 are mutually exclusive.

Now $\quad P(\text{a total of } 9) = P[(3, 6), (4, 5), (5, 4), (6, 3)] = \dfrac{4}{36}$

$$P(\text{a total of } 11) = P[(5, 6), (6, 5)] = \frac{2}{36}$$

Thus, $\quad P(\text{a total of 9 or } 11) = \dfrac{4}{36} + \dfrac{2}{36} = \dfrac{1}{6}$

Example 19.10 Prove that the probability of the non-occurrence of an event A is $1 - P(A)$.

i.e., $\quad P(\text{not } A) = 1 - P(A) \quad$ or, $\quad P(\bar{A}) = 1 - P(A)$.

Solution : We know that the probability of the sample space S in any experiment is 1.

Now, it is clear that if in an experiment an event A occurs, then the event $(\bar{A})$ cannot occur simultaneously, i.e., the two events are mutually exclusive.

Also, the sample points of the two mutually exclusive events together constitute the sample space S. That is,

$$A \cup \bar{A} = S$$

Thus, $\quad P\left(A \cup \bar{A}\right) = P(S)$

$$\Rightarrow P(A) + P(\bar{A}) = 1 \;(\because A \text{ and } \bar{A} \text{ are mutually exclusive and S is sample space})$$

$$\Rightarrow P\left(\bar{A}\right) = 1 - P(A),$$

which proves the result.

This is called the law of complementation.

Law of complimentation : $P\left(\bar{A}\right) = 1 - P(A)$

$$\frac{P(\bar{A})}{P(A)} \text{ or } P(A) \text{ to } P(\bar{A}).$$

Example 19.11 The probability of the event that it will rain is 0.3. Find the odds in favour of rain and odds against rain.

Solution : Let A be the event that it will rain. $\therefore$ $P(A) = .3$

By law of complementation, $P\left(\bar{A}\right) = 1 - .3 = .7$.

Now, the odds in favour of rain are $\frac{0.3}{0.7}$ or 3 to 7 (or 3 : 7).

The odds against rain are $\frac{0.7}{0.3}$ or 7 to 3.

When either the odds in favour of A or the odds against A are given, we can obtain the probability of that event by using the following formulae

If the odds in favour of A are *a* to *b*, then $P(A) = \frac{a}{a+b}$.

If the odds against A are a to b, then $P(A) = \frac{b}{a+b}$.

This can be proved very easily.

Suppose the odds in favour of A are *a* to *b*. Then, by the definition of odds,

$$\frac{P(A)}{P(\bar{A})} = \frac{a}{b}.$$

From the law of complimentation, $P\left(\bar{A}\right) = 1 - P(A)$

Therefore, $\quad \frac{P(A)}{1 - P(A)} = \frac{a}{b} \quad$ or $\quad b\,P(A) = a - a\,P(A)$

or $\quad (a+b)\,P(A) = a \quad$ or $\quad P(A) = \frac{a}{a+b}$

Similarly, we can prove that $P(A) = \frac{b}{a+b}$

when the odds against A are b to a.

Example 19.12 Are the following probability assignments consistent ? Justify your answer.

(a) $\quad P(A) = P(B) = 0.6, \qquad P(A \text{ and } B) = 0.05$

(b) $\quad P(A) = 0.5, \quad P(B) = 0.4, \quad P(A \text{ and } B) = 0.1$

(c) $\quad P(A) = 0.2, \quad P(B) = 0.7, \quad P(A \text{ and } B) = 0.4$

Solution : (a) $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

$$= 0.6 + 0.6 - 0.05 = 1.15$$

Since P (A or B) > 1 is not possible, hence the given probabilities are not consistent.

(b) $$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$
$$= 0.5 + 0.4 - 0.1 = 0.8$$

which is less than 1.

As the number of outcomes favourable to event 'A and B' should always be less than or equal to those favourable to the event A,

Therefore, $\quad P(A \text{ and } B) \leq P(A)$

and similarly $\quad P(A \text{ and } B) \leq P(B)$

In this case, P (A and B) = 0.1, which is less than both P (A) = 0.5 and P (B) = 0.4. Hence, the assigned probabilities are consistent.

(c) In this case, P (A and B) = 0.4, which is more than P (A) = 0.2.

$$[\because P(A \text{ and } B) \leq P(A)]$$

Hence, the assigned probabilities are not consistent.

Example 19.13 An urn contains 8 white balls and 2 green balls. A sample of three balls is selected at random. What is the probability that the sample contains at least one green ball ?

Solution : Urn contains 8 white balls and 2 green balls.

$\therefore$ Total number of balls in the urn = 10

Three balls can be drawn in ${}^{10}C_3$ ways = 120 ways.

Let A be the event " at least one green ball is selected".

Let us determine the number of different outcomes in A. These outcomes contain either one green ball or two green balls.

There are ${}^{2}C_1$ ways to select a green ball from 2 green balls and for this remaining two white balls can be selected in ${}^{8}C_2$ ways.

Hence, the number of outcomes favourable to one green ball

$$= {}^{2}C_1 \times {}^{8}C_2 = 2 \times 28 = 56$$

Similarly, the number of outcomes favourable to two green balls

$$= {}^{2}C_2 \times {}^{8}C_1 = 1 \times 8 = 8$$

Hence, the probability of at least one green ball is

P (at least one green ball)

$$= P(\text{one green ball}) + P(\text{two green balls})$$

$$= \frac{56}{120} + \frac{8}{120} = \frac{64}{120} = \frac{8}{15}$$

Example 19.14 Two balls are drawn at random with replacement from a bag containing 5 blue and 10 red balls. Find the probability that both the balls are either blue or red.

Solution : Let the event A consists of getting both blue balls and the event B is getting both red balls. Evidently A and B are mutually exclusive events.

By fundamental principle of counting, the number of outcomes favourable to $A = 5 \times 5 = 25.$

Similarly, the number of outcomes favourable to B = 10 × 10 = 100.

Total number of possible outcomes = 15 × 15 = 225.

$$\therefore \qquad P(A) = \frac{25}{225} = \frac{1}{9} \quad \text{and} \quad P(B) = \frac{100}{225} = \frac{4}{9}.$$

Since the events A and B are mutually exclusive, therefore

$$P(A \text{ or } B) = P(A) + P(B)$$

$$= \frac{1}{9} + \frac{4}{9} = \frac{5}{9}$$

Thus, P (both blue or both red balls) = $\frac{5}{9}$

CHECK YOUR PROGRESS 19.3

1. A card is drawn from a well-shuffled pack of cards. Find the probability that it is a queen or a card of heart.
2. In a single throw of two dice, find the probability of a total of 7 or 12.
3. The odds in favour of winning of Indian cricket team in 2010 world cup are 9 to 7. What is the probability that Indian team wins ?
4. The odds against the team A winning the league match are 5 to 7. What is the probability that the team A wins the league match.
5. Two dice are thrown. Getting two numbers whose sum is divisible by 4 or 5 is considered a success. Find the probability of success.
6. Two cards are drawn at random from a well-shuffled deck of 52 cards with replacement. What is the probability that both the cards are either black or red ?
7. A card is drawn at random from a well-shuffled deck of 52 cards. Find the probability that the card is an ace or a black card.
8. Two dice are thrown once. Find the probability of getting a multiple of 3 on the first die or a total of 8.
9. (a) In a single throw of two dice, find the probability of a total of 5 or 7.

   (b) A and B are two mutually exclusive events such that P (A) = 0.3 and P (B) = 0.4. Calculate P (A or B).
10. A box contains 12 light bulbs of which 5 are defective. All the bulbs look alike and have equal probability of being chosen. Three bulbs are picked up at random. What is the probability that at least 2 are defective ?
11. Two dice are rolled once. Find the probability

(a) that the numbers on the two dice are different,

(b) that the total is at least 3.

12. A couple have three children. What is the probability that among the children, there will be at least one boy or at least one girl ?

13. Find the odds in favour and against each event for the given probability

(a) P (A) = .7 (b) P (A) = $\frac{4}{5}$

14. Determine the probability of A for the given odds

(a) 7 to 2 in favour of A (b) 10 to 7 against A.

15. If two dice are thrown, what is the probability that the sum is

(a) greater than 4 and less than 9 ?

(b) neither 5 nor 8 ?

16. Which of the following probability assignments are inconsistent ? Give reasons.

(a) P (A) = 0.5 , P (B) = 0.3 , P (A and B) = 0.4

(b) P (A) = P (B) = 0.4, P (A and B) = 0.2

(c) P (A) = 0.85, P (B) = 0.8, P (A and B) = 0.61

17. Two balls are drawn at random from a bag containing 5 white and 10 green balls. Find the probability that the sample contains at least one white ball.

18. Two cards are drawn at random from a well-shuffled deck of 52 cards with replacement. What is the probability that both cards are of the same suit?

Thus, the probability of simultaneous occurrence of two independent events is the product of their separate probabilities.

19.6 MULTIPLICATION LAW OF PROBABILITY FOR INDEPENDENT EVENTS

Let us recall the definition of independent events.

Two events A and B are said to be independent, if the occurrence or non-occurrence of one does not affect the probability of the occurrence (and hence non-occurrence) of the other.

Can you think of some examples of independent events ?

The event of getting 'H' on first coin and the event of getting 'T' on the second coin in a simultaneous toss of two coins are independent events.

What about the event of getting 'H' on the first toss and event of getting 'T' on the second toss in two successive tosses of a coin ? They are also independent events.

Let us consider the event of 'drawing an ace' and the event of 'drawing a king' in two successive draws of a card from a well-shuffled deck of cards without replacement.

Are these independent events ?

No, these are not independent events, because we draw an ace in the first draw with probability

$\frac{4}{52}$. Now, we do not replace the card and draw a king from the remaining 51 cards and this affect the probability of getting a king in the second draw, i.e., the probability of getting a king in the second draw without replacement will be $\frac{4}{51}$.

Note : If the cards are drawn with replacement, then the two events become independent.

Is there any rule by which we can say that the events are independent ?

How to find the probability of simultaneous occurrence of two independent events?

If A and B are independent events, then

$$P\ (A\ \text{and}\ B) = P\,(A)\ .\ P\,(B)$$

or

$$P\,(A \cap B) = P\,(A).\ P\,(B)$$

Thus, the probability of simultaneous occurrence of two independent events is the product of their separate probabilities.

Note : The above law can be extended to more than two independent events, i.e.,

$$P(A \cap B \cap C...) = P(A) \cdot P(B) \cdot P(C)...$$

On the other hand, if the probability of the event 'A' and 'B' is equal to the product of the probabilities of the events A and B, then we say that the events A and B are independent.

Example 19.15 A die is tossed twice. Find the probability of a number greater than 4 on each throw.

Solution : Let us denote by A, the event 'a number greater than 4' on first throw. B be the event 'a number greater than 4' in the second throw. Clearly A and B are independent events.

In the first throw, there are two outcomes, namely, 5 and 6 favourable to the event A.

$\therefore$ $$P(A) = \frac{2}{6} = \frac{1}{3}$$

Similarly, $$P\,(B) = \frac{1}{3}$$

Hence, $P\,(A\ \text{and}\ B) = P\ (A)\ .\ P\ (B) = \frac{1}{3}.\frac{1}{3} = \frac{1}{9}$.

Example 19.16 Arun and Tarun appear for an interview for two vacancies. The probability of Arun's selection is $\frac{1}{3}$ and that of Tarun's selection is $\frac{1}{5}$. Find the probability that

(a) both of them will be selected. (b) none of them is selected.

(c) at least one of them is selected. (d) only one of them is selected.

Solution : Probability of Arun's selection $= P(A) = \frac{1}{3}$

Probability of Tarun's selection $= P(T) = \frac{1}{5}$

(a) P (both of them will be selected) = P (A) P (T)

$$= \frac{1}{3} \times \frac{1}{5} = \frac{1}{15}$$

(b) P (none of them is selected)

$$= P\left(\bar{A}\right) P\left(\bar{T}\right) = \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right) = \frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$$

(c) P (at least one of them is selected)

$$= 1 - P \text{ (None of them is selected)}$$

$$= 1 - P\left(\bar{A}\right) P\left(\bar{T}\right) = 1 - \left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)$$

$$= 1 - \left(\frac{2}{3} \times \frac{4}{5}\right) = 1 - \frac{8}{15} = \frac{7}{15}$$

(d) P (only one of of them is selected)

$$= P(A) P\left(\bar{T}\right) + P\left(\bar{A}\right) P(T)$$

$$= \frac{1}{3} \times \frac{4}{5} + \frac{2}{3} \times \frac{1}{5} = \frac{6}{15} = \frac{2}{5}$$

Example 19.17 A problem in statistics is given to three students, whose chances of solving it are $\frac{1}{2}, \frac{1}{3}$ and $\frac{1}{4}$ respectively. What is the probability that problem will be solved ?

Solution : Let p_1, p_2 and p_3 be the probabilities of three persons of solving the problem.

Here, $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{3}$ and $p_3 = \frac{1}{4}$.

The problem will be solved, if at least one of them solves the problem.

$\therefore$ P (at least one of them solves the problem)

$$= 1 - P \text{ (None of them solves the problem)} \qquad .....(1)$$

Now, the probability that none of them solves the problem will be

P (none of them solves the problem) $= (1 - p_1)(1 - p_2)(1 - p_3)$

$$= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$$

Putting this value in (1), we get

$$\text{P (at least one of them solves the problem)} = 1 - \frac{1}{4} = \frac{3}{4}$$

Hence, the probability that the problem will be solved is $\frac{3}{4}$.

Example 19.18 Two balls are drawn at random with replacement from a box containing 15 red and 10 white balls. Calculate the probability that

(a) both balls are red.

(b) first ball is red and the second is white.

(c) one of them is white and the other is red.

Solution :

(a) Let A be the event that first drawn ball is red and B be the event that the second ball drawn is red. Then as the balls drawn are with replacement,

therefore $\quad P(A) = \frac{15}{25} = \frac{3}{5}, \; P(B) = \frac{3}{5}$

As A and B are independent events

therefore $\quad$ P (both red) = P (A and B)

$$= P(A) \times P(B) = \frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$$

(b) Let A : First ball drawn is red.

B : Second ball drawn is white.

$$\therefore \quad P(\text{A and B}) = P(A) \times P(B) = \frac{3}{5} \times \frac{2}{5} = \frac{6}{25}.$$

(c) If WR denotes the event of getting a white ball in the first draw and a red ball in the second draw and the event RW of getting a red ball in the first draw and a white ball in the second draw. Then as 'RW' and WR' are mutually exclusive events, therefore

$\therefore \quad$ P (a white and a red ball)

$$= P(\text{WR or RW})$$

$$= P(WR) + P(RW)$$

$$= P(W)\,P(R) + P(R)\,P(W)$$

$$= \frac{2}{5} \cdot \frac{3}{5} + \frac{3}{5} \cdot \frac{2}{5}$$

$$= \frac{6}{25} + \frac{6}{25} = \frac{12}{25}$$

Example 19.19 A dice is thrown 3 times. Getting a number '5 or 6' is a success. Find the probability of getting

(a) 3 successes (b) exactly 2 successes (c) at most 2 successes (d) at least 2 successes.

Solution : Let S denote the success in a trial and F denote the ' not success' i.e. failure. Therefore,

$$P(S) = \frac{2}{6} = \frac{1}{3}, \quad P(F) = 1 - \frac{1}{3} = \frac{2}{3}$$

(a) As the trials are independent, by multiplication theorem for independent events,

$$P(SSS) = P(S)\ P(S)\ P(S) = \frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$$

$$P(SSF) = P(S)\,P(S)\,P(F) = \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{27}$$

Since the two successes can occur in ${}^3C_2$ ways

$$\therefore \quad P(\text{exactly two successes}) = {}^3C_2 \times \frac{2}{27} = \frac{2}{9}$$

(c) $P(\text{ at most two successes}) = 1 - P(3\text{ successes}) = 1 - \frac{1}{27} = \frac{26}{27}$

(d) $P(\text{ at least two successes}) = P(\text{ exactly 2 successes}) + P(\text{ 3 successes})$

$$= \frac{2}{9} + \frac{1}{27} = \frac{7}{27}$$

Example 19.20 A card is drawn from a pack of 52 cards so that each card is equally likely to be selected. Which of the following events are independent ?

(i) A : the card drawn is a spade

B : the card drawn is an ace

(ii) A : the card drawn is black

B : the card drawn is a king

(iii) A : the card drawn is a king or a queen

B : the card drawn is a queen or a jack

Solution : (i) There are 13 cards of spade in a pack. $P(A) = \frac{13}{52} = \frac{1}{4}$

There are four aces in the pack. $P(B) = \frac{4}{52} = \frac{1}{13}$

$$A \cap B = \{ \text{an ace of spade} \}$$

$\therefore$ $$P(A \cap B) = \frac{1}{52}$$

Now $$P(A) \; P(B) = \frac{1}{4} \times \frac{1}{13} = \frac{1}{52}$$

Since $$P(A \cap B) = P(A).\ P(B)$$

Hence, the events A and B are independent.

(ii) There are 26 black cards in a pack.

$\therefore$ $$P(A) = \frac{26}{52} = \frac{1}{2}$$

There are four kings in the pack. $\therefore$ $$P(B) = \frac{4}{52} = \frac{1}{13}$$

$$A \cap B = \{ \text{2 black kings} \} \; \therefore \; P(A \cap B) = \frac{2}{52} = \frac{1}{26}$$

Now, $$P(A) \times P(B) = \frac{1}{2} \times \frac{1}{13} = \frac{1}{26}$$

Since $$P(A \cap B) = P(A) \; P(B)$$

Hence, the events A and B are independent.

(iii) There are 4 kings and 4 queens in a pack of cards.

$\therefore$ Total number of outcomes favourable to the event A is 8.

$\therefore$ $$P(A) = \frac{8}{52} = \frac{2}{13}$$

Similarly, $$P(B) = \frac{2}{13}, \quad A \cap B = \{ \text{4 queens} \}$$

$\therefore$ $$P(A \cap B) = \frac{4}{52} = \frac{1}{13}$$

$\therefore$ $$P(A) \times P(B) = \frac{2}{13} \times \frac{2}{13} = \frac{4}{169}$$

Here, $$P(A \cap B) \neq P(A).P(B)$$

Hence, the events A and B are not independent.

CHECK YOUR PROGRESS 19.4

1. A husband and wife appear in an interview for two vacancies in the same department. The probability of husband's selection is $\frac{1}{7}$ and that of wife's selection is $\frac{1}{5}$. What is the probability that

(a) Only one of them will be selected ?

(b) Both of them will be selected ?

(c) None of them will be selected ?

(d) At least one of them will be selected ?

2. Probabilities of solving a specific problem independently by Raju and Soma are $\frac{1}{2}$ and $\frac{1}{3}$ respectively. If both try to solve the problem independently, find the probability that

(a) the problem is solved.

(b) exactly one of them solves the problem.

3. A die is rolled twice. Find the probability of a number greater than 3 on each throw.

4. Sita appears in the interview for two posts A and B, selection for which are independent. The probability of her selection for post A is $\frac{1}{5}$ and for post B is $\frac{1}{7}$. Find the probability that she is selected for

(a) both the posts

(b) at least one of the posts.

5. The probabilities of A, B and C solving a problem are $\frac{1}{3}, \frac{2}{7}$ and $\frac{3}{8}$ respectively. If all the three try to solve the problem simultaneously, find the probability that exactly one of them will solve it.

6. A draws two cards with replacement from a well-shuffled deck of cards and at the same time B throws a pair of dice. What is the probability that

(a) A gets both cards of the same suit and B gets a total of 6 ?

(b) A gets two jacks and B gets a doublet ?

7. Suppose it is 9 to 7 against a person A who is now 35 years of age living till he is 65 and 3:2 against a person B now 45 living till he is 75. Find the chance that at least one of these persons will be alive 30 years hence.

8. A bag contains 13 balls numbered from 1 to 13. Suppose an even number is considered a 'success'. Two balls are drawn with replacement, from the bag. Find the probability of getting

(a) Two successes (b) exactly one success

(c) at least one success (d) no success

9. One card is drawn from a well-shuffled deck of 52 cards so that each card is equally likely to be selected. Which of the following events are independent ?

(a) A : The drawn card is red

B : The drawn card is a queen

(b) A: The drawn card is a heart B: The drawn card is a face card

19.7 CONDITIONAL PROBABILITY

Suppose that a fair die is thrown and the score noted. Let A be the event, the score is 'even'. Then

$$A = \{2, 4, 6\}, \therefore \; P(A) = \frac{3}{6} = \frac{1}{2}.$$

Now suppose we are told that the score is greater than 3. With this additional information what will be P (A) ?

Let B be the event, 'the score is greater than 3'. Then B is {4, 5, 6 }. When we say that B has occurred, the event 'the score is less than or equal to 3' is no longer possible. Hence the sample space has changed from 6 to 3 points only. Out of these three points 4, 5 and 6; 4 and 6 are even scores.

Thus, given that B has occurred, P (A) must be $\frac{2}{3}$.

Let us denote the probability of A given that B has already occurred by $P(A \mid B)$.

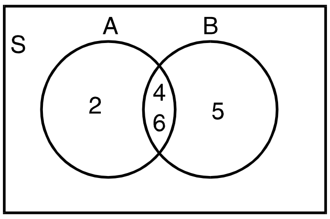


FIg. 19.7

Again, consider the experiment of drawing a single card from a deck of 52 cards. We are interested in the event A consisting of the outcome that a black ace is drawn.

Since we may assume that there are 52 equally likely possible outcomes and there are two black aces in the deck, so we have

$$P(A) = \frac{2}{52} \quad .$$

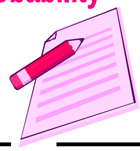
However, suppose a card is drawn and we are informed that it is a spade. How should this information be used to reappraise the likelihood of the event A ?

Clearly, since the event B "A spade has been drawn " has occurred, the event "not spade" is no longer possible. Hence, the sample space has changed from 52 playing cards to 13 spade cards. The number of black aces that can be drawn has now been reduced to 1.

Therefore, we must compute the probability of event A relative to the new sample space B.

Let us analyze the situation more carefully.

The event A is " a black ace is drawn'. We have computed the probability of the event A knowing that B has occurred. This means that we are computing a probability relative to a new sample space B. That is, B is treated as the universal set. We should consider only that part of A which is included in B.



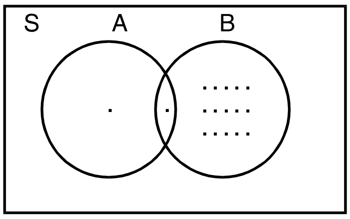


Fig. 19.8

Hence, we consider $A \cap B$ (see figure 31.8).

Thus, the probability of A. given B, is the ratio of the number of entries in $A \cap B$ to the number of entries in B. Since $n(A \cap B) = 1$ and $n(B) = 13$,

then $$P(A \mid B) = \frac{n(A \cap B)}{n(B)} = \frac{1}{13}$$

Notice that $$n(A \cap B) = 1 \Rightarrow P(A \cap B) = \frac{1}{52}$$

$$n(B) = 13 \Rightarrow P(B) = \frac{13}{52}$$

$$\therefore \quad P(A \mid B) = \frac{1}{13} = \frac{\frac{1}{52}}{\frac{13}{52}} = \frac{P(A \cap B)}{P(B)}.$$

This leads to the definition of conditional probability as given below :

Let A an B be two events defined on a sample space S. Let P(B) > 0, then the conditional probability of A, provided B has already occurred, is denoted by P(A|B) and mathematically written as :

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}, \quad P(B) > 0$$

Similarly, $$P(B \mid A) = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0$$

The symbol P(A | B) is usually read as "the probability of A given B".

Example 19.21 Consider all families "with two children (not twins). Assume that all the elements of the sample space {BB, BG, GB,GG} are equally likely. (Here, for instance, BG denotes the birth sequence "boy girls"). Let A be the event {BB} and B be the event that 'at least one boy'. Calculate P (A | B).

Solution : Here, $A = \{BB\}$, $B = \{BB, BG, GB\}$

$$A \cap B = \{BB\} \therefore \quad P(A \cap B) = \frac{1}{4}$$

$$P(B) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

Hence, $$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

Example 19.22 Assume that a certain school contains equal number of female and male students. 5 % of the male population is football players. Find the probability that a randomly selected student is a football player male.

Solution : Let M = Male

F = Football player

We wish to calculate $P(M\cap F)$. From the given data,

$$P(M) = \frac{1}{2} \quad (\because \text{ School contains equal number of male and female students})$$

$$P(F|M) = 0.05$$

But from definition of conditional probability, we have

$$P(F|M) = \frac{P(M\cap F)}{P(M)}$$

$$\Rightarrow \qquad P(M\cap F) = P(M)\times P(F|M)$$

$$= \frac{1}{2}\times 0.05 = 0.025$$

Example 19.23 If A and B are two events, such that $P(A) = 0.8$, $P(B) = 0.6$, $P(A\cap B) = 0.5$, find the value of

(i) $P(A\cup B)$ (ii) $P(B|A)$ (iii) $P(A|B)$.

Solution :(i) $$P(A\cup B) = P(A) + P(B) - P(A\cap B)$$

$$= 0.8 + 0.6 - 0.5 = 0.9$$

(ii) $$P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{0.5}{0.8} = \frac{5}{8}$$

(iii) $$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{0.5}{0.6} = \frac{5}{6}$$

Example 19.24 A coin is tossed until a head appears or until it has been tossed three times. Given that head does not occur on the first toss, what is the probability that coin is tossed three times ?

Solution : Here, it is given that head does not occur on the first toss. That is, we may get the head on the second toss or on the third toss or even no head.

Let B be the event, " no heads on first toss".

Then $B = \{TH, TTH, TTT\}$

These events are mutually exclusive.

$$P(B) = P(TH) + P(TTH) + P(TTT) \qquad .....(1)$$

Now $P(TH) = \frac{1}{4}$ ($\because$ This event has the sample space of four outcomes)

and $P(TTH) = P(TTT) = \frac{1}{8}$ ($\because$ This event has the sample space of eight outcomes)

Putting these values in (1), we get

$$P(B) = \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

Let A be the event "coin is tossed three times".

Then $A = \{TTH, TTT\}$

$\therefore$ We have to find $P(A|B)$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Here, $A \cap B = A$, $\therefore$ $P(A|B) = \dfrac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$

CHECK YOUR PROGRESS 19.5

1. A sequence of two cards is drawn at random (without replacement) from a well-shuffled deck of 52 cards. What is the probability that the first card is red and the second card is black ?
2. Consider a three child family for which the sample space is

   { BBB, BBG, BGB, GBB, BGG, GBG, GGB, GGG }

   Let A be the event " the family has exactly 2 boys " and B be the event " the first child is a boy". What is the probability that the family has 2 boys, given that first child is a boy ?
3. Two cards are drawn at random without replacement from a deck of 52 cards. What is the probability that the first card is a diamond and the second card is red ?
4. If A and B are events with $P(A) = 0.4$, $P(B) = 0.2$, $P(A \cap B) = 0.1$, find the probability of A given B. Also find $P(B|A)$.
5. From a box containing 4 white balls, 3 yellow balls and 1 green ball, two balls are drawn one at a time without replacement. Find the probability that one white and one yellow ball is drawn.

19.8 THEOREMS ON MULTIPLICATION LAW OF PROBABILITY AND CONDITIONAL PROBABILITY.

Theorem 1 : For two events A and B,

$$P(A \cap B) = P(A).P(B \mid A),$$

and
$$P(A \cap B) = P(B).P(A \mid B),$$

where $P(B|A)$ represents the conditional probability of occurrence of B, when the event A has already occurred and $P(A|B)$ is the conditional probability of happening of A, given that B has already happened.

Proof : Let n (S) denote the total number of equally likely cases, n (A) denote the cases favourable to the event A, n (B) denote the cases favourable to B and $n(A \cap B)$ denote the cases favourable to both A and B.

$$\therefore \qquad P(A) = \frac{n(A)}{n(S)}, \quad P(B) = \frac{n(B)}{n(S)}$$

$$P(A \cap B) = \frac{n(A \cap B)}{n(S)} \qquad .....(1)$$

For the conditional event A|B , the favourable outcomes must be one of the sample points of B, i.e., for the event A|B, the sample space is B and out of the n (B) sample points, $n(A \cap B)$ pertain to the occurrence of the event A, Hence,

$$P(A \mid B) = \frac{n(A \cap B)}{n(B)}$$

Rewriting (1), we get $\quad P(A \cap B) = \frac{n(B)}{n(S)} \cdot \frac{n(A \cap B)}{n(B)} = P(B).P(A|B)$

Similarly, we can prove

$$P(A \cap B) = P(A).P(B \mid A)$$

Note : If A and B are independent events, then

$$P(A \mid B) = P(A) \text{ and } P(B \mid A) = P(B)$$

$$P(A \cap B) = P(A).P(B)$$

Theorem 2 : Two events A and B of the sample space S are independent, if and only if

$$P(A \cap B) = P(A).P(B)$$

Proof : If A and B are independent events,

then $\qquad P(A \mid B) = P(A)$

We know that $\qquad P(A \mid B) = \frac{P(A \cap B)}{P(B)}$

$$\Rightarrow \qquad P(A \cap B) = P(A)P(B)$$

Hence, if A and B are independent events, then the probability of 'A and B' is equal to the product of the probability of A and probability of B.

Conversely, if $P(A \cap B) = P(A).P(B)$, then

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} \text{ gives}$$

$$P(A \mid B) = \frac{P(A)P(B)}{P(B)} = P(A)$$

That is, A and B are independent events.

19.9 INTRODUCTION TO BAYES' THEOREM

In conditional probability we have learnt to find probability of an event with the condition that some other event has already occurred. Consider an experiment of selecting one coin out of three coins : If I with P(H) = $\frac{1}{3}$ and P(T) = $\frac{2}{3}$, II with P(H) = $\frac{3}{4}$ and P(T) = $\frac{1}{4}$ and III with P(H) = $\frac{1}{2}$, P(T) = $\frac{1}{2}$ (a normal coin).

After randomly selecting one of the coins, it is tossed. We can find the probability of selecting one coin $\left(\text{i.e. } \frac{1}{3}\right)$ and can also find the probability of any outcome i.e. head or tail; given the coin selected. But can we find the probability that coin selected is coin I, II or III when it is known that the head occurred as outcome? For this we have to find the probability of an event which occurred prior to the given event. Such probability can be obtained by using Bayes' theorem, named after famous mathematician, Johan Bayes Let us first learn some basic definition before taking up Baye's theorem

Mutually exclusive and exhaustive events.

For a sample space S, the set of events $E_1, E_2, \ldots E_n$ is said to mutually exclusive and exhaustive if

(i) $E_i \cap E_j = \phi, \forall i \neq j = 1, 2, \ldots\ldots n$ i.e. none of two events can occur together.

(ii) $E_i \cup E_2 \cup \ldots\ldots \cup E_n = S$, all outcomes of S have been taken up in the events E_1, $E_2 \ldots E_n$

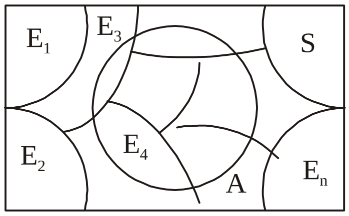
(iii) $P(E_i) > 0$ for all $i = 1, 2, \ldots\ldots n$

19.10 : THEOREM OF TOTAL PROBABILITY

Let $E_1, E_2, \ldots., E_n$ are mutually exclusive and exhaustive events for a sample space S with $P(E_i) > 0, \quad \forall i = 1, 2, \ldots n$. Let A be any event associated with S, then

$$P(A) = P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + \ldots + P(E_n)\, P(A/E_n)$$

$$= \sum_{i=1}^{n} P(E_i)P(A/E_i)$$



$U(A \cap E_n)$

Fig. 19.9

Proof : The events E_i and A are shown in the venn-diagram

Given $S = E_1 \cup E_2, \cup E_3, \ldots \cup E_n$ and $E_i \cap E_j \neq \phi$.

We can write

$$\begin{aligned} A &= A \cap S \\ &= A \cup (E_1 \cup E_2 \cup \ldots \cup E_n) \\ &= (A \cap E_1) \cap (A \cap E_2) \cup (A \cap E_3) \ldots (A \cap E_n) \end{aligned}$$

Since all E_i, are mutually exclusive, so $A \cap E_1$, $A \cap E_2$ ... will also be mutually exclusive

$$\Rightarrow \quad P(A) = P(A \cap E_1) + P(A \cap E_2) + P(A \cap E_3) + \ldots + P(A \cap E_n)$$
$$= P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + \ldots + P(E_n)\, P(A/En)$$

By using the multiplication rule of probability,

$$P(A) = \sum_{i=1}^{n} P(E_i)P(A/E_i)$$

19.11 : BAYE'S THEOREM

If $E_1, E_2, \ldots E_n$ are non-empty mutually exclusive and exhaustive events (i.e. $P(E_i) > 0 \ \forall\, i$) of a sample space S and A be any event of non-zero probability then

$$P(Ei/A) = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A/E_i)} \ \forall\ i = 1, 2, \ldots n$$

Proof : By law of total probabilities we know that

$$P(A) = \sum_{i=1}^{n} P(E_i)\, P(A/E_i) \qquad \ldots(i)$$

Also by law of multiplication of probabilities we have

$$P(E_i|A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)P(A/E_i)} \text{ by using (i)}$$

This gives the proof of the Baye's theorem let us now apply the result of Baye's theorem to find probabilities.

Example 19.25 Given three identical coins (in shape and size) with following specifications

Coin I : with $P(H) = \frac{1}{3}$, $P(T) = \frac{2}{3}$

Coin II : with $P(H) = \frac{3}{4}$ $P(T) = \frac{1}{4}$

Coin III : with $P(H) = \frac{1}{2}$, $P(T) = \frac{1}{2}$ (normal coin).

A Coin is selected at random and tossed. The out come found to be head. What is the probability that the selected coin was coin III?

Solution : Let E_1, E_2, E_3 be the events that coins I, II or III is selected, respectively.

Then $\quad P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Also, Let A be the event 'the coin drawn 'has head on tossing'.

Then $\quad P(A/E_1) = P(\text{a head on coin I}) = \frac{1}{3}$

$P(A/E_2) = P(\text{a head on coin II}) = \frac{3}{4}$

$P(A/E_3) = P(\text{a head on coin III}) = \frac{1}{2}$

Now the probability that the coin tossed is Coin III = $P(E_3/A)$

$$= \frac{P(E_3)\,P(A/E_3)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)}$$

$$= \frac{\frac{1}{3}\times\frac{1}{2}}{\frac{1}{3}\times\frac{1}{3}+\frac{1}{3}\times\frac{3}{4}+\frac{1}{3}\times\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{3}+\frac{3}{4}+\frac{1}{2}} = \frac{6}{4+9+6} = \frac{6}{19}$$

Example 19.26 Bag I contains 4 red and 3 black balls while another bag II contains 6 red and 5 black balls. One of the bags is selected at random and a ball is drawn from it. Find the probability that the ball is drawn from Bag II, if it is known that the ball drawn is red.

Solution : Let E_1 and E_2 be the events of selecting Bag I and Bag II, respectively and A be the event of selecting a red ball.

Then, $\quad P(E_1) = P(E_2) = \frac{1}{2}$

Also, $\quad P(A/E_1) = P(\text{drawing a red ball from Bag I}) = \frac{4}{7}$

$P(A/E_2) = P(\text{drawing a red ball from Bag II}) = \frac{6}{11}$

Now, By Baye's theorem

P(bag selected is Bag II when it is known that red ball is drawn) = $P(E_2/A)$

$$= \frac{P(E_2)\,P(A/E_2)}{P(E_1)\,P(A/E_1)+P(E_2)P(A/E_2)} = \frac{\frac{1}{2}\times\frac{4}{7}}{\frac{1}{2}\times\frac{6}{11}+\frac{1}{2}\times\frac{4}{7}}$$

$$= \frac{4/7}{\frac{6}{11}+\frac{4}{7}} = \frac{22}{43}$$

CHECK YOUR PROGRESS 19.6

1. Urn I contains 3 blue and 4 white balls and another Urn II contains 4 blue and 3 white balls. One Urn was selected at random and a ball was drawn from the selected Urn. The ball was found to be white. What is the probability that the ball was drawn from Urn-II?
2. A factory has two machines A and B. Past record shows that machine A produced 60% of the items of output and machine B produced 40% of the items. Further, 2% of the items produced by machine A and 1% by machine B were defective. All the items are put in one stock pile and then one item is randomly drawn from this and is found to be defective. Find the probability that the defective item was produced by machine A?
3. By examining the chest x-ray, the probability that T.B is detected when a person is actually suffering from it is 0.99.

   The probability that the doctor, diagnoses in correctly that a person has TB, on the basis of the x-ray is 0.001. In a certain city, 1 in 10000 persons suffer from TB. A person selected at random is diagnosed to have TB. What is the probability that person has actually TB?

19.12 : PROBABILITY DISTRIBUTION OF RANDOM VARIABLE

19.12.1 Variables : In earlier section you have learnt to find probabilities of various events with certain conditions. Let us now consider the case of tossing a coin four times. The outcomes can be shown in a sample space as :

S = {HHHH, HHHT, HHTH, HTHH, THHH, THHT, HHTT, HTTH, TTHH, HTHT, THTH, HTTT, THTT, TTHT, TTTH, TTTT}

On this sample space we can talk about various number associated with each outcome. For example, for each outcome, there is a number corresponding to number of heads we can call this number as X.

Clearly

X(HHHH) = 4, X(HHHT) = 3, X(HHTH) = 3

X(THHH) = 3, X(HHTT) = 2, X(HTTH) = 2

X(TTHH) = 2, X(HTHT) = 2, X(THTH) = 2

X(THHT) = 2, X(HTTT) = 1, X(THTT) = 1

X(TTHT) = 1, X(TTTH) = 1, X(TTTT) = 0

We find for each out come there corresponds values of X ranging from 0 to 4.

Such a variable X is called a random variable.

19.12.2 Definition

A random variable is a function whose domain is the sample space of a random experiment and range is real number values.

Example 19.27 Two dice are thrown simultaneously. Write the value of the random variable X : sum of number appearing on the upper faces of the dice.

Solution : The sample space of the experiment contains 36 elements.

S = {(1, 1), (1, 2), (1, 3) .... (1, 6)

(2, 1), (2, 2), (2, 3) .... (2, 6)

....

....

....

(6, 1), (6, 2), (6, 3) .... (6, 6)}

Clearly for each pair the sum of numbers appear ranging from 2 to 12. So the random variable x has the following values.

X(1, 1) = 2

X((1, 2), (2, 1)) = 3

X((1, 3), (2, 2), (3, 1)) = 4

X((1, 4), (2, 3), (3, 2), (4, 1) = 5

X((1, 5), (2, 4), (3, 3), (4, 2), (5, 1) = 6

X((1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1), = 7

X((2, 6), (3, 5), (4, 4), (5, 3), (6, 2) = 8

X((3, 6), (4, 5), (5, 4), (6, 3) = 9

X((4, 6), (5, 5), (6, 4) = 10

X((5, 6), (6, 5)) = 11

X((6, 6)) = 12

19.12.3 Probability Distribution of a Random Variable

Let us now look at the experiment of drawing two cards successively with replacement from a well shuffled deck of 52 cards. Let us concentrate on the number of aces that can be there when two cards are successively drawn. Let it be denoted by X. Clearly X can take the values 0, 1 or 2.

The sample space for the experiment is given by S = {(Ace, Ace), (Ace, Non Ace), (Non Ace, Ace), (Non Ace, Non Ace)}

For X(Ace, Ace) = 2

X{(Ace, Non Ace) or (Non Ace, Ace)} = 1

and X{(Non Ace, Non Ace)} = 0

The probability that X can take the value 2 is P(Ace, Ace) = $\frac{4}{52}\times\frac{4}{52}$ as probability of an Ace is drawing one card is $\frac{4}{52}$.

Similarly

$$P(X = 1) = P[(\text{Ace, non Ace}) \text{ or } (\text{Non Ace, ace})]$$
$$= P(\text{Ace, non Ace}) + P(\text{Non Ace, Ace})$$
$$= \frac{4}{52}\times\frac{48}{52}+\frac{48}{52}\times\frac{4}{52} = \frac{12}{169}+\frac{2}{169}=\frac{24}{169}$$

and $$P(X = 0) = P(\text{Non Ace, Non Ace}) = \frac{48}{52}\times\frac{48}{52}=\frac{144}{169}$$

The description given by the values of the random variable with the corresponding probabilities is called probability distribution.

19.12.4 Definition : The probability distribution of a random variable X is the distribution of probabilities to each value of X. A probability distribution of a random variable X is represented as

X_i	:	x_1	x_2	x_3	....	x_n
$P(X_i)$	:	P_1	P_2	P_3	....	P_n

where $P_i > 0, \sum_{i=1}^{n} P_i = 1, \forall\ i = 1, 2, 3, \ldots n$.

The real numbers $x_1, x_2, \ldots x_n$ are the possible values of X and P_i is the probability of the random variable X taking the value X_i denoted as

$$P(X = x_i) = P_i$$

Thus the probability distribution of number of aces when two cards are successively drawn, with replacement from a deck of 52 cards is given by

X	:	0	1	2
$P(x_i)$	:	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

Note that in a probability distribution all probabilities must be between 0 and 1 and sum of all probabilities must be 1.

$$\Sigma P_i = \frac{144}{169}+\frac{24}{169}+\frac{1}{169}=\frac{144+24+1}{169}=1$$

Example 19.28 Check whether the distribution given below is a probability distribution or not

X	2	1	0	–1	–2
P(X)	0.1	0.2	0.3	0.2	0.2

Solution : All probabilities P(X) are positive and less than 1.

Also,
$$\Sigma P(x_i) = 0.1 + 0.2 + 0.3 + 0.2 + 0.2$$
$$= 1.0$$

Hence, the given distribution is probability distribution of a the random variable X.

Example 19.29 A random variable X has the following probability distribution :

X	–1	–2	–3	–4	–5	–6
P(X)	$\frac{1}{3}$	k	$\frac{1}{4}$	$2k$	$\frac{1}{6}$	$\frac{k}{4}$

Find (1) k (2) $P(X > -4)$ (3) $P(X < -4)$

Solution : (1) The sum of probabilities in the given distribution, must be 1.

$$\Rightarrow \quad \frac{1}{3}+k+\frac{1}{4}+2k+\frac{1}{6}+\frac{k}{4} = 1$$

$$\Rightarrow \quad \frac{4+12k+3+24k+2+3k}{12} = 1$$

$$39k + 9 = 12$$

$$\Rightarrow \quad 39k = 3$$

$$\therefore \quad k = \frac{1}{13}$$

(2)
$$P(X > -4) = P(x = -3) + P(x = -2) + P(x = -1)$$
$$= \frac{1}{4}+k+\frac{1}{3} = \frac{1}{4}+\frac{1}{13}+\frac{1}{3} = \frac{103}{156}$$

(3)
$$P(X < -4) = P(x = -5) + P(x = -6)$$
$$= \frac{1}{6}+\frac{k}{4} = \frac{1}{6}+\frac{1}{13\times 4} = \frac{29}{156}$$

Example 19.30 Find the probability distribution of number of tails in the simultaneous tosses of three coins.

Solution : The sample space for simultaneous toss of three coins is given by

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

Let X be the number of tails.

Clearly X can take values, 0, 1, 2 or 3.

Now,

$$P(X=0) = P(HHH) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$$

$$P(X=1) = P(HHT \text{ or } HTH \text{ or } THH)$$

$$= P(HHT) + P(HTH) + P(THH)$$

$$= \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$$

$$P(X=2) = P(HTT \text{ or } THT \text{ or } TTH)$$

$$= P(HTT) + P(THT) + P(TTH) = \frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{3}{8}$$

and $P(X=3) = P(TTT) = \frac{1}{8}$.

Hence, the required probability distribution is

X	:	0	1	2	3
P(X)	:	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

CHECK YOUR PROGRESS 19.7

1. State which of the following are not probability distribution of a random variable. Justify your answer

(a)

x	100	200	300
P(x)	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

(b)

Y	0	1	2	3	4	5
P(y)	0.1	0.2	0.3	0.4	0.5	0.6

(c)

x_i	–1	–2	0	2	1
P_i	0.2	0.15	–0.5	0.45	0.7

(d)

x_i	2	3	4	5
P_i	0.4	0.1	0.2	0.2

2. Find the probability distribution of

(a) Number of red balls when two balls drawn are one after other with replacement from a bag containing 4 red and 3 white balls.

(b) Number of sixes when two dice are thrown simultaneously

(c) Number of doublets when two dice are thrown simultaneously

19.13 : MEAN AND VARIANCE OF A RANDOM VARIABLE

19.13.1 Mean

The mean of a random variable is denoted by μ and is defined as

$$\mu = \sum_{i=1}^{n} x_i P_i \,,$$

where $\Sigma P_i = 1$, $P_i > 0$, $\forall\, i = 1, 2, \ldots n$.

In other words we can say that the mean of a random variable is the sum of the product of values of the variables with corresponding probabilities. Mean of a random variable X is also called Expectation of the random variable 'X', denoted by E(x)

So $$E(x) = \mu = \sum_{i=1}^{n} x_i P_i \,.$$

19.13.2 : VARIANCE

Recall in frequently distribution we have studied that variance is a measure of dispersion or variability in the values. The similar meaning is attached to variance of a random variable.

Definition : Let a probability distribution be given as

X_i	:	x_1	x_2	x_3	...	x_n
$P(X_i)$	:	P_1	P_2	P_3	...	P_n

Let $\mu = E(x)$ be the mean of x.

Then the varaiance of X, denoted by var(x) or $\sigma_x^{\,2}$ is defined as

$$\sigma_x^{\,2} = \text{Var}(x) = \sum_{i=1}^{n} (x_i - \mu)^2 P_i$$

$$= \sum_{i=1}^{n} (x_1^2 p_i + \mu^2 p_i - 2\mu\, x_i p_i) = \sum_{i=1}^{n} (x_i^2 P_i + \mu^2 P_i - 2\mu x_i\; p_i)$$

$$= \sum_{i=1}^{n} x_i^2 p_i + \sum_{i=1}^{n} \mu^2 P_i - \sum_{i=1}^{n} 2\mu x_i\; p_i = \sum_{i=1}^{n} x_i^2 p_i + \mu^2 \sum_{i=1}^{n} P_i - 2\mu \sum_{i=1}^{n} x_i\; p_i$$

$$= \sum_{i=1}^{n} x_i^2 p_i + \mu^2 .1 - 2\,\mu.\mu = \sum_{i=1}^{n} x_i^2 p_i - \mu^2 \quad (\because\; \mu = \sum_{i=1}^{n} x_i^2 p_i \text{ and } \sum_{i=1}^{n} p_i = 1)$$

$$= \sum_{i=1}^{n} x_i^2 p_i - \left(\sum_{i=1}^{n} x_i p_i \right)^2$$

We can also write

$$\text{var}(x) = E(X^2) - [E(X)]^2$$

Example 19.31 Find the mean and variance of the following distribution

x	-2	-1	0	1	2
$P(x)$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$

Solution : Given distribution is

X_i	-2	-1	0	1	2
$P(X_i)$	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$X_iP(X_i)$	$-\frac{2}{8}$	$-\frac{2}{8}$	0	$\frac{1}{8}$	$\frac{2}{8}$
$x^2P(X_i)$	$\frac{4}{8}$	$\frac{1}{8}$	0	$\frac{1}{8}$	$\frac{4}{8}$

Now,

$$\mu = \Sigma P(X_i)X_i = -\frac{2}{8} - \frac{2}{8} + 0 + \frac{1}{8} + \frac{2}{8} = -\frac{1}{8}$$

$$\text{Var}(x) = \Sigma X_i^2 P(X_i) - [\Sigma\, P(X_i)X_i]^2$$

$$= \left[\frac{4}{8} + \frac{2}{8} + 0 + \frac{1}{8} + \frac{4}{8}\right] - \left(-\frac{1}{8}\right)^2$$

$$= \frac{11}{8} - \frac{1}{64} = \frac{87}{64}$$

CHECK YOUR PROGRESS 19.8

1. Find mean and variance in each of the following distributions

(a)

X	:	1	2	3	4
P(X)	:	0.3	0.2	0.4	0.1

(b)

y_i	-2	-1	0	1	2
$P(y_i)$	0.1	0.2	0.3	0.25	0.15

2. Find the mean number of heads in three tosses of a fair coin.
3. Let X denote the difference of two numbers obtained on throwing two fair dice. Find the mean and variance of X. (Take absolute value of the difference)
4. Find the mean of the numbers of tails obtained when a biased coin having 25% chances of head and 75% of tail, is tossed two times.
5. Find the mean and variance of the number of sixes when two dice are thrown.

19.14 BERNOULLI TRIALS

When an experiment is repeated under similar conditions, each repeat is called a trial of the experiment. For example, if a coin is tossed three times, we say that there are three trials of the tossing of the coin.

A particular event may be called success of a trial. Clearly non-happening of the event may be termed as a failure. For, example in throwing a die, if the occurrence of a number less then 4 is named as success then the non-occurrence of a number less than 4 is named as failure. Thus, each trial can have two outcomes namely, success or failure.

Two or more trials of a random experiment can be performed in two ways :

1. The probability of success or failure remain constant in each trial. For example tossing a coin n number of times, but in each trial probability of getting head is $\frac{1}{2}$. Such trials are called independent trials.
2. The probability of success/failure varies with each trial. For example in drawing card from a deck of cards one after the other without replacement, in such trials if success is taken to be drawing a card of spade, the probability of success in respective trials will change.

i.e.	Trial	1st	2nd	3rd, ...
	Probability	$\frac{13}{52}$	$\frac{12}{51}$	$\frac{11}{50}$, ...

The trials of first type i.e. independent trials with two out comes success or failure are called Bernoulli trials.

Definition : Trials of a random experiment are called Bernoulli trials, if each trial has exactly two outcomes and trials are finite and independent.

19.15 : BINOMIAL DISTRIBUTION

The probability distribution of number successes in Bernoulli trials of a random experiment may be obtained by the expansion of $(q + p)^n$ where

p = prob. of success in each trial

q = $1 - p$, = prob. of failure

n = number of trials

Such a probability distribution is called Binomial Distribution. In other words we can say that in n Bernoulli trials of a random experiment, the number of successes can have the value, 0, 1, 2, 3, ...n.

So the Binomial Distribution of number of success : X, is given by

$$P(X = 0) = \text{1st term of the expansion of } (q + p)^n$$

$$P(X= 1) = \text{2nd term of the expansion of } (q + p)^n$$

$$\vdots$$

$$P(X = r) = (r + 1)\text{th term of expansion of } (q + p)^n$$

$$\vdots$$

$$P(X = n) = (n + 1)\text{th term of expansion of } (q + p)n$$

We know that

$$(q + p)^n = {}^nC_0\, q^n + {}^nC_1\, q^{n-1}\, p + {}^nC_2\, q^{n-2} p^2 + ... + {}^nC_r\, q^{n-r} p^r + ... + {}^nC_n p^n$$

$$\Rightarrow \quad P(x = 0) = {}^nC_0\, q^n$$

$$P(x = 1) = {}^nC_1\, q^{n-1} p$$

$$P(x = 2) = {}^nC_2\, q^{n-2} p^2$$

$$\vdots$$

$$P(X = r) = {}^nC_r\, q^{n-r} p^r$$

$$\vdots$$

$$P(X = n) = {}^nC_n\, p^n.$$

A Binomial distribution with n Bernoulli trials and probability of success in each trial as P, is denoted by B(n, p)

Let us now understand Binomial Distribution with following examples.

Example 19.32 Write the Binomial Distribution of number of successes in 3 Bernoulli trials.

Solution : Let p = prob. of success (S) in each trial

q = prob. of failure (F) in each trial

Clearly $q = 1 - p$

Number of successes in three trials can take the values 0, 1, 2 or 3

The sample space for three trials ....

$$S = \{SSS, SSF, SFS, FSS, SFF, FSF, FFS, FFF\}$$

where S and F denote success and failure.

Now $P(S = 0) = P(FFF) = P(F)\ P(F)\ P(F) = q.q.q = q^3$

$$P(S = 1) = P(SFF, FSF \text{ or } FFS) = P(SFF) + P(FSF) + P(FFS)$$

$$= P(S).P(F)\ P(F) + P(F).P(S).P(F) + P(F)\ P(F)\ P(S)$$

$$= p.q.q. + q.p.q + q.q.p = 3\ q^2 p$$

$$
\begin{aligned}
P(S = 2) &= P(SSF \text{ or } SFS \text{ or } FSS) \\
&= P(SSF) + P(SFS) + P(FSS) \\
&= P(S).P(S).P(F) + P(S)\,P(F)P(S) + P(F)\,P(S)\,P(S) \\
&= p.p.q + p.q.p + q.p.p = 3qp^2 \\
P(S = 3) &= P(SSS) = P(S)\,.\,P(S)\,.\,P(S) = p.p.p = p^3
\end{aligned}
$$

Hence the prob. distribution of number of successes is

X_i	:	0	1	2	3
$P(X_i)$	:	q^3	$3q^2p$	$3qp^2$	p^3

Also $(q + p)^3 = q^3 + 3q^2p + 3p^2q + p^3$

Note that probabilities of 0, 1, 2 or 3 successes are respectively the 1st, 2nd, 3rd and 4th term in the expansion of $(q + p)^3$.

Example 19.33 A die is thrown 5 times. If getting 'an even number' is a success, what is the probability of.

(a) 5 successes

(b) at least 4 successes

(c) at most 3 successes?

Solution : Given X : "an even number"

Then p =P(an even number) $= \frac{3}{6} = \frac{1}{2}$

$$q = P(\text{not an even number}) = \frac{3}{6} = \frac{1}{2}$$

Since the trials of throwing die are Bernoulli trials.

So, $\qquad P(r \text{ successes}) = {}^nC_r\, q^{n-r}\, p^r$

Here, $n = 5 = {}^5C_r \left(\frac{1}{2}\right)^{5-2} \left(\frac{1}{2}\right)^2 = {}^5C_r \left(\frac{1}{2}\right)^5$

(a) Now P(5 successes) $= {}^5C_5 \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

(b) P(at least 3 successes)

= P(3 success or 4 successes or 5 successes)

= P(3 successes) + P(4 successes) + P(5 successes).

$$= {}^5C_3 \left(\frac{1}{2}\right)^5 + {}^5C_4 \left(\frac{1}{2}\right)^5 + {}^5C_5 \left(\frac{1}{2}\right)^5$$

$$= \left(\frac{1}{2}\right)^5 \left(\frac{5\times4\times3}{3\times2\times1}+5+1\right) = \left(\frac{1}{2}\right)^5 (10+5+1) = \frac{16}{32} = \frac{1}{2}$$

(c) P(at most 3 successes)

= P(0 successes or 1 success or 2 success or 3 successes)

= P(0 successes) + P(1 success) + P(2 successes) + P(3 successes)

$$= {}^5C_0\left(\frac{1}{2}\right)^5 + {}^5C_1\left(\frac{1}{2}\right)^5 + {}^5C_2\left(\frac{1}{2}\right)^5 + {}^5C_3\left(\frac{1}{2}\right)^5$$

$$= \frac{1}{32}+5\times\frac{1}{32}+\frac{5\times4}{2\times1}\times\frac{1}{32}+\frac{5\times4\times3}{3\times2\times1}\times\frac{1}{32}$$

$$= \frac{1}{32}[1+5+10+10] = \frac{26}{32} = \frac{13}{16}.$$

CHECK YOUR PROGRESS 19.9

1. Find the following probabilities when a fair coin is tossed 10 times.
   (a) exactly 6 heads
   (b) at least 6 heads
   (c) at most 6 heads
2. A pair of dice is thrown 4 times. If getting a doublet (1, 1), (2, 2)... etc. is considered a success, find the probability of two successes.
3. From a bag containing 3 red and 4 black sells, five balls are drawn successively with replacement. If getting "a black ball" is considered "success", find the probability of getting 3 successes.
4. In a lot of bulbs manufactured in a factory, 5% are defective. What is the probability that a sample of 10 bulbs will include not more than one defective bulb?
5. Probability that a CFL produced by a factory will fuse after 1 year of use is 0.01. Find the probability that out of 5 such CFL's.
   (a) none
   (b) not more than one
   (c) more than one
   (d) at least one

   will fuse after 1 year of use.

LET US SUM UP

- **Complement of an event :** The complement of an event A consists of all those outcomes which are not favourable to the event A, and is denoted by 'not A' or by $\bar{A}$.

- **Event 'A or B' :** The event 'A or B' occurs if either A or B or both occur.
- **Event 'A and B' :** The event 'A and B' consists of all those outcomes which are favourable to both the events A and B.
- **Addition Law of Probability :** For any two events A and B of a sample space S

  P (A or B) = P (A) + P (B) – P (A and B)

- **Additive Law of Probability for Mutually Exclusive Events :** If A and B are two mutually exclusive events, then

  P (A or B) = P (A ∪ B) = P (A) + P (B).

- **Odds in Favour of an Event :** If the odds for A are a to b, then $P(A) = \dfrac{a}{a+b}$

  If odds against A are a to b, then $P(A) = \dfrac{b}{a+b}$

- Two events are mutually exclusive, if occurrence of one precludes the possibility of simultaneous occurrence of the other.
- Two events are independent, if the occurence of one does not affect the occurence of other. If A and B are independent events, then P (A and B) = P (A). P (B) or $P(A \cap B) = P(A).P(B)$
- For two dependent events $P(A \cap B) = P(A).P\left(B/A\right) where\, P(A) > 0$

  or

  $P(A \cap B) = P(B) where\, P\left(A/B\right) / P(B) > 0$

- Conditional Probability $P\left(\dfrac{A}{B}\right) = \dfrac{A(A \cap B)}{P(B)}$ and $P\left(B/A\right) = \dfrac{P(A \cap B)}{P(A)}$
- Theorem of Total Probability

  $$P(A) = P(E_1).\,P\left(\frac{A}{E_1}\right) + P(E_2).\,P\left(\frac{A}{E_2}\right) + \cdots + P(E_n).P\left(\frac{A}{E_n}\right)$$

- Baye's Theorem: If $B_1, B_2 \cdots B_n$ are mutually exclusive events and A is any event that occurs with B_1 or B_2 or B_n then $P\left(\dfrac{Bi}{A}\right) = \dfrac{P(Bi), P\left(A/Bi\right)}{\sum_{i=1}^{n} P(Bi), P\left(A/Bi\right)}, i = 1, 2, \cdots n$
- Mean and Variance of a Random Varibale

  $$\mu = E(x) = \frac{n}{z} XiPi\,,\ \sigma^2 = \frac{n}{z}(xi - \mu)^2 = \frac{n}{z} xi^2 pi - \mu^2$$

- Binomial Distribution, $P(x = r) = nc_r\, p^r .q^{n-r}$

SUPPORTIVE WEB SITES

http://en.wikipedia.org/wiki/Probability

http://mathworld.wolfram.com/Probability

http://en.wikipedia.org/wiki/probability _distribution

http://en.wikipedia.org/wiki/Probability_theory

http://en.wikipedia.org/wiki/Bernoulli_distribution

TERMINAL EXERCISE

1. In a simultaneous toss of four coins, what is the probability of getting
   (a) exactly three heads ?
   (b) at least three heads ?
   (c) atmost three heads ?
2. Two dice are thrown once. Find the probability of getting an odd number on the first die or a sum of seven.
3. An integer is chosen at random from first two hundred integers. What is the probability that the integer chosen is divisible by 6 or 8 ?
4. A bag contains 13 balls numbered from 1 to 13. A ball is drawn at random. What is the probability that the number obtained it is divisible by either 2 or 3 ?
5. Find the probability of getting 2 or 3 heads, when a coin is tossed four times.
6. Are the following probability assignments consistent ? Justify your answer.
   (a) P (A) = 0.6, P (B) = 0.5 , P (A and B) = 0.4
   (b) P (A) = 0.2, P (B) = 0.3 , P (A and B) = 0.4
   (c) P (A) = P (B) = 0.7, P (A and B) = 0.2
7. A box contains 25 tickets numbered 1 to 25. Two tickets are drawn at random. What is the probability that the product of the numbers is even ?
8. A drawer contains 50 bolts and 150 nuts. Half of the bolts and half of the nuts are rusted. If one item is chosen at random, what is the probability that it is rusted or is a bolt ?
9. A lady buys a dozen eggs, of which two turn out to be bad. She chose four eggs to scramble for breakfast. Find the chances that she chooses
   (a) all good eggs
   (b) three good and one bad eggs
   (c) two good and two bad eggs (d) at least one bad egg.

10. Two cards are drawn at random without replacement from a well-shuffled deck of 52 cards. Find the probability that the cards are both red or both kings.

11. Let A and B be two events such that $P(\vec{A})=\frac{1}{2}, P(\vec{B})=\frac{2}{3}, P(A\overset{\rightarrow}{\cap}B)=\frac{1}{4}$, Find $P(A/B)$ and $P(B/A)\cdot$

12. A bag contains 10 black and 5 white balls. Two balls are drawn from the bag successively whithout replacement. Find the probability that both the balls drawn are black.

13. Find the probability distribution of X; where X denotes the sum of numbers obtained when two dice are rolled.

14. An urn contains 4 black, 2 red and 2 white balls. Two balls (one after the other without replacement) are drawn randomly from the urn. Find the probability distribution of number of black balls.

15. Find the mean and variance of number of kings when two cards are simultaneously drawn from a deck of 52 cards.

16. Ten bolts are drawn successively with replacement from a bag containing 5% defective bolts. Find the probability that there is at least one defective bolt.

17. Find the mean of the Binomial $B\left(4,\frac{1}{3}\right)$.

18. A die is thrown again and again until three sixes are obtained. Find the probability of getting the third six in the sixth throw.

19. How many times must a man toss a fair coin so that the probability of having atleast one head is more than 90%?

20. Find the probability of getting 5 exactly twice in seven throws of a die.

21. Find the mean number of heads in three tosses of a fair coin.

22. A factory produces nuts, by using three machine A, B and C, manufacturing 20%, 40% and 40% of the nuts. 5%, 4% and 2% of their outputs are respectively found to be defective, A nut is drawn of randomly from the product and is found to be defective. What is the probability that it is manufactured by the machine C?

ANSWERS

CHECK YOUR PROGRESS 19.1

1. $\frac{1}{6}$ 2. $\frac{1}{2}$ 3. $\frac{1}{2}$ 4. $\frac{3}{4}$

5. (i) $\frac{3}{5}$ (ii) $\frac{2}{5}$

6. (i) $\frac{5}{36}$ (ii) $\frac{5}{36}$ (iii) $\frac{1}{12}$ (iv) $\frac{1}{36}$

7. $\frac{5}{9}$ 8. $\frac{1}{12}$ 9. $\frac{1}{2}$

10. (i) $\frac{1}{4}$ (ii) $\frac{1}{13}$ (iii) $\frac{1}{52}$

11. (i) $\frac{5}{12}$ (ii) $\frac{1}{6}$ (iii) $\frac{11}{36}$

12. (i) $\frac{1}{8}$ (ii) $\frac{7}{8}$ (iii) $\frac{1}{8}$

CHECK YOUR PROGRESS 19.2

1. $\frac{1}{8}$ 2. $\frac{20}{39}$ 3.(a) $\frac{4}{25}$ (b) $\frac{38}{245}$

4. $\frac{1}{5525}$ 6. (i) $\frac{3}{10}$ (ii) $\frac{1}{6}$ (iii) $\frac{1}{30}$

7. $\frac{10}{133}$ 8. $\frac{4}{7}$ 9. $\frac{60}{143}$ 10. $\frac{1}{4}$

CHECK YOUR PROGRESS 19.3

1. $\frac{4}{13}$ 2. $\frac{7}{36}$ 3. $\frac{9}{16}$ 4. $\frac{7}{12}$

5. $\frac{4}{9}$ 6. $\frac{1}{2}$ 7. $\frac{7}{13}$ 8. $\frac{5}{12}$

9. (a) $\frac{5}{18}$ (b) 0.7 10. $\frac{4}{11}$

11. (a) $\frac{5}{6}$ (b) $\frac{35}{36}$ 12. $\frac{3}{4}$

13. (a) The odds for A are 7 to 3 . The odds against A are 3 to 7

13. (b) The odds for A are 4 to 1 and The odds against A are 1 to 4

14. (a) $\frac{7}{9}$ (b) $\frac{7}{17}$ 15. (a) $\frac{5}{9}$ (b) $\frac{3}{4}$ 16. (a), (c) 17. $\frac{4}{7}$ 18. $\frac{1}{4}$

CHECK YOUR PROGRESS 19.4

1. (a) $\frac{2}{7}$ (b) $\frac{1}{35}$ (c) $\frac{24}{35}$ (d) $\frac{11}{35}$ 2. (a) $\frac{2}{3}$ (b) $\frac{1}{2}$ 3. $\frac{1}{4}$

4. (a) $\frac{1}{35}$ (b) $\frac{11}{35}$ 5. $\frac{1}{2}$ 6. (a) $\frac{5}{144}$ (b) $\frac{1}{1014}$ 7. $\frac{53}{80}$

8. (a) $\frac{36}{169}$ (b) $\frac{84}{169}$ (c) $\frac{120}{169}$ (d) $\frac{49}{169}$

9. (a) Independent (b) Independent

CHECK YOUR PROGRESS 19.5

1. $\frac{13}{51}$ 2. $\frac{1}{2}$ 3. $\frac{25}{204}$ 4. $\frac{1}{2}, \frac{1}{4}$ 5. $\frac{3}{7}$

CHECK YOUR PROGRESS 19.6

1. $\frac{3}{7}$ 2. $\frac{3}{4}$ 3. $\frac{10}{111}$

CHECK YOUR PROGRESS 19.7

1. (a) Yes (b) No, as ΣP_i is not 1

(c) No, as one of the P_i is –ve (d) No as ΣP_i is not 1

2. (a)

x	:	0	1	2
$P(x)$	:	$\frac{9}{49}$	$\frac{24}{49}$	$\frac{16}{49}$

(b)

X_i	:	0	1	2
$P(X_i)$	:	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

CHECK YOUR PROGRESS 19.8

1. (a) $\mu = 2.3$, Var $= 1.01$ (b) $\mu = 0.15$, Var $= 0.4275$

2. $\mu = \frac{3}{2}$

3.

X	:	0	1	2	3	4	5
$P(X_i)$	:	$\frac{6}{36}$	$\frac{10}{36}$	$\frac{8}{36}$	$\frac{6}{36}$	$\frac{4}{36}$	$\frac{2}{36}$

4. Mean $\mu = \frac{3}{2}$, Var. $(X_i) = \frac{3}{8}$ 5. Mean $= \frac{1}{3}$, Var. $= \frac{5}{18}$

CHECK YOUR PROGRESS 19.9

1. (i) $\frac{105}{512}$ (ii) $\frac{193}{512}$ (iii) $\frac{53}{64}$

2. $\frac{25}{216}$ 3. $\frac{90 \times 64}{7^5}$ 4. $\left(\frac{29}{20}\right)\left(\frac{19}{20}\right)^9$

5. (a) $\left(\frac{99}{100}\right)^5$ (b) $\left(\frac{99}{100}\right)^5 + 5.\frac{99^4}{100^5}$

(c) $1-\left\{\left(\frac{99}{100}\right)^5 + \frac{5 \times 99^4}{100^5}\right\}$ (d) $1-\left(\frac{99}{100}\right)^5$

TERMINAL EXERCISE

1. (a) $\frac{1}{4}$ (b) $\frac{5}{16}$ (c) $\frac{15}{16}$ 2. $\frac{7}{12}$ 3. $\frac{1}{4}$

4. $\frac{8}{13}$ 5. $\frac{5}{8}$ 6. Only (a) is consistent 7. $\frac{456}{625}$

8. $\frac{5}{8}$ 9. (a) $\frac{14}{33}$ (b) $\frac{16}{33}$ (c) $\frac{1}{11}$ (d) $\frac{19}{33}$

10. $\frac{55}{221}$ 11. $\frac{3}{4}, \frac{1}{2}$ 12. $\frac{3}{7}$

13.

X_i	:	2	3	4	5	6	7	8	9	10	11	12
$P(X_i)$	:	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

14.

x	:	0	1	2
$P(x)$	:	$\frac{3}{14}$	$\frac{4}{7}$	$\frac{3}{14}$

15. Mean $= \frac{34}{221}$, variance $= \frac{6800}{(221)^2}$. 16. $1-\left(\frac{19}{20}\right)^{10}$ 17. $\frac{4}{3}$

18. $\frac{625}{23328}$ 19. $n = 4$ 20. $\frac{7}{12} \times \left(\frac{5}{6}\right)^5$

21. 1.5 22. $\frac{4}{17}$