211en22

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INTRODUCTION TO TRIGONOMETRY

Study of triangles occupies important place in Mathematics. Triangle being the bounded figure with minimum number of sides serve the purpose of building blocks for study of any figure bounded by straight lines. Right angled triangles get easy link with study of circles as well.

In Geometry, we have studied triangles where most of the results about triangles are given in the form of statements. Here in trigonometry, the approach is quite different, easy and crisp. Most of the results, here, are the form of formulas. In Trigonometry, the main focus is study of right angled triangle. Let us consider some situations, where we can observe the formation of right triangles.

Have you seen a tall coconut tree? On seeing the tree, a question about its height comes to the mind. Can you find out the height of the coconut tree without actually measuring it? If you look up at the top of the tree, a right triangle can be imagined between your eye, the top of the tree, a horizontal line passing through the point of your eye and a vertical line from the top of the tree to the horizontal line.

Let us take another example.

Suppose you are flying a kite. When the kite is in the sky, can you find its height? Again a right triangle can be imagined to form between the kite, your eye, a horizontal line passing through the point of your eye, and a vertical line from the point on the kite to the horizontal line.

Let us consider another situation where a person is standing on the bank of a river and observing a temple on the other bank of the river. Can you find the width of the river if the height of the temple is given? In this case also you can imagine a right triangle.

Finally suppose you are standing on the roof of your house and suddenly you find an aeroplane in the sky. When you look at it, again a right triangle can be imagined. You find the aeroplane moving

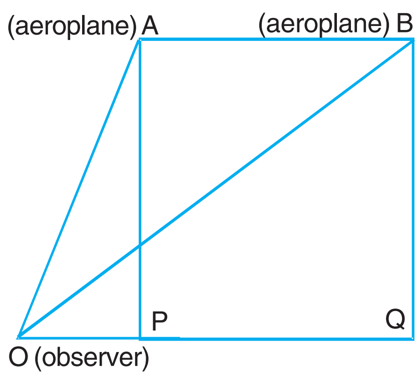


Fig. 22.1

away from you and after a few seconds, if you look at it again, a right triangle can be imagined between your eye, the aeroplane and a horizontal line passing through the point (eye) and a vertical line from the plane to the horizontal line as shown in the figure.

Can you find the distance AB, the aeroplane has moved during this period?

In all the four situations discussed above and in many more such situations, heights or distance can be found (without actually measuring them) by using some mathematical techniques which come under branch of Mathematics called, "Trigonometry".

Trigonometry is a word derived from three Greek words- 'Tri' meaning 'Three' 'Gon' meaning 'Sides' and 'Metron' meaning 'to measure'. Thus Trigonometry literally means measurement of sides and angles of a triangle. Originally it was considered as that branch of mathematics which dealt with the sides and the angles of a triangle. It has its application in astronomy, geography, surveying, engineering, navigation etc. In the past astronomers used it to find out the distance of stars and planets from the earth. Now a day, the advanced technology used in Engineering is based on trigonometrical concepts.

In this lesson, we shall define trigonometric ratios of angles in terms of ratios of sides of a right triangle and establish relationship between different trigonometric ratios. We shall also establish some standard trigonometric identities.

OBJECTIVES

After studying this lesson, you will be able to

- *write the trigonometric ratios of an acute angle of right triangle;*
- *find the sides and angles of a right triangle when some of its sides and trigonometric ratios are known;*
- *write the relationships amongst trigonometric ratios;*
- *establish the trigonometric identities;*
- *solve problems based on trigonometric ratios and identities;*
- *find trigonometric ratios of complementary angles and solve problems based on these.*

EXPECTED BACKGROUND KNOWLEDGE

- Concept of an angle
- Construction of right triangles
- Drawing parallel and perpendiculars lines

- Types of angles- acute, obtuse and right
- Types of triangles- acute, obtuse and right
- Types of triangles- isosceles and equilateral
- Complementary angles.

22.1 TRIGONOMETRIC RATIOS OF AN ACUTE ANGLE OF A RIGHT ANGLED TRIANGLE

Let there be a right triangle ABC, right angled at B. Here ∠A (i.e. ∠CAB) is an acute angle, AC is hypotenuse, side BC is opposite to ∠A and side AB is adjacent to ∠A.

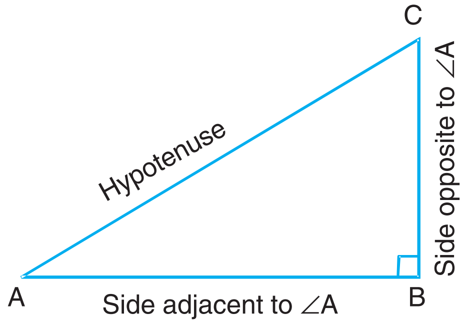


Fig. 22.2

Again, if we consider acute ∠C, then side AB is side opposite to ∠C and side BC is adjacent to ∠C.

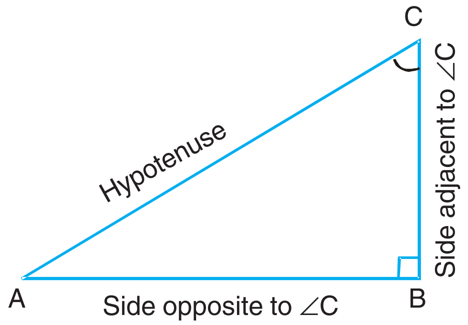


Fig. 22.3

We now define certain ratios involving the sides of a right triangle, called **trigonometric ratios**.

The trigonometric ratios of ∠A in right angled ΔABC are defined as:

(i) $\text{sine } A = \dfrac{\text{side opposite to } \angle A}{\text{Hypotenuse}} = \dfrac{BC}{AC}$

(ii) $\text{cosine } A = \dfrac{\text{side adjacent to } \angle A}{\text{Hypotenuse}} = \dfrac{AB}{AC}$

(iii) $\text{tangent } A = \dfrac{\text{side opposite to } \angle A}{\text{side adjacent to } \angle A} = \dfrac{BC}{AB}$

(iv) $\text{cosecant } A = \dfrac{\text{Hypotenuse}}{\text{side opposite to } \angle A} = \dfrac{AC}{BC}$

(v) $\text{secant } A = \dfrac{\text{Hypotenuse}}{\text{side adjacent to } \angle A} = \dfrac{AC}{AB}$

(vi) $\text{cotangent } A = \dfrac{\text{side adjacent to } \angle A}{\text{side opposite to } \angle A} = \dfrac{AB}{BC}$

The above trigonometric ratios are abbreviated as sin A, cos A, tan A, cosec A, sec A and cot A respectively. Trigonometric ratios are abbreviated as **t-ratios**.

If we write $\angle A = \theta$, then the above results are

$$\sin\theta = \frac{BC}{AC}, \qquad \cos\theta = \frac{AB}{AC}, \qquad \tan\theta = \frac{BC}{AB}$$

$$\text{cosec}\,\theta = \frac{AC}{BC}, \qquad \sec\theta = \frac{AC}{AB} \qquad \text{and} \quad \cot\theta = \frac{AB}{BC}$$

Note: Observe here that sin θ and cosec θ are reciprocals of each other. Similarly cot θ and sec θ are respectively reciprocals of tan θ and cos θ.

Remarks

Thus in right ΔABC,

AB = 4cm, BC = 3cm and

AC = 5cm, then

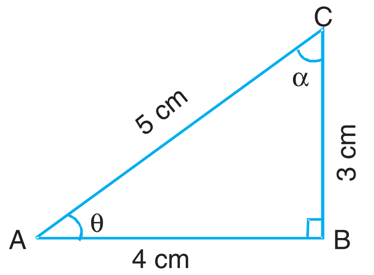


Fig. 22.4

$$\sin\theta = \frac{BC}{AC} = \frac{3}{5}$$

$$\cos\theta = \frac{AB}{AC} = \frac{4}{5}$$

$$\tan\theta = \frac{BC}{AB} = \frac{3}{4}$$

$$\text{cosec}\,\theta = \frac{AC}{BC} = \frac{5}{3}$$

$$\sec\theta = \frac{AC}{AB} = \frac{5}{4}$$

and $$\cot\theta = \frac{AB}{BC} = \frac{4}{3}$$

In the above figure, if we take angle C = α, then

$$\sin\alpha = \frac{\text{side opposite to } \angle\alpha}{\text{Hypotenuse}} = \frac{AB}{AC} = \frac{4}{5}$$

$$\cos\alpha = \frac{\text{side adjacent to } \angle\alpha}{\text{Hypotenuse}} = \frac{BC}{AC} = \frac{3}{5}$$

$$\tan\alpha = \frac{\text{side opposite to } \angle\alpha}{\text{side adjacent to } \angle\alpha} = \frac{AB}{BC} = \frac{4}{3}$$

$$\text{cosec}\,\alpha = \frac{\text{Hypotenuse}}{\text{side opposite to } \angle\alpha} = \frac{AC}{AB} = \frac{5}{4}$$

$$\sec\alpha = \frac{\text{Hypotenuse}}{\text{side adjacent to } \angle\alpha} = \frac{AC}{BC} = \frac{5}{3}$$

and $$\cot\alpha = \frac{\text{side adjacent to } \angle\alpha}{\text{side opposite to } \angle\alpha} = \frac{BC}{AB} = \frac{3}{4}$$

Remarks :

1. Sin A or sin θ is one symbol and sin cannot be separated from A or θ. It is not equal to sin × θ. The same applies to other trigonometric ratios.
2. Every t-ratio is a real number.
3. For convenience, we use notations $\sin^2\theta$, $\cos^2\theta$, $\tan^2\theta$ for $(\sin\theta)^2$, $(\cos\theta)^2$, and $(\tan\theta)^2$ respectively. We apply the similar notation for higher powers of trigonometric ratios.
4. We have restricted ourselves to t-ratios when A or θ is an acute angle.

Now the question arises: "Does the value of a t-ratio remains the same for the same angle of different right triangles?." To get the answer, let us consider a right triangle ABC, right angled at B. Let P be any point on the hypotenuse AC.

Let PQ ⊥ AB

Now in right ΔABC,

$$\sin A = \frac{BC}{AC} \qquad \text{----(i)}$$

and in right ΔAQP,

$$\sin A = \frac{PQ}{AP} \qquad \text{----(ii)}$$

Now in ΔAQP and ΔABC,

$\angle Q = \angle B$ ----(Each = 90°)

and $\angle A = \angle A$ ----(Common)

$\therefore$ $\Delta AQP \sim \Delta ABC$

$$\therefore \frac{AP}{AC} = \frac{QP}{BC} = \frac{AQ}{AB}$$

or $$\frac{BC}{AC} = \frac{PQ}{AP} \qquad \text{----(iii)}$$

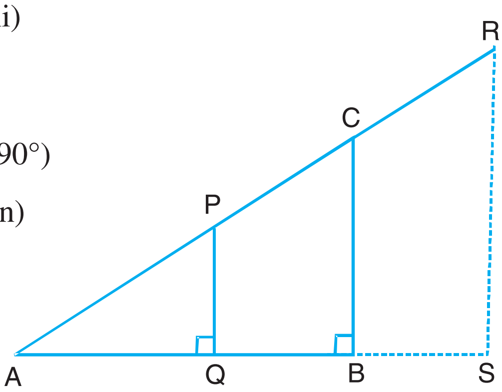


Fig. 22.5

From (i), (ii), and (iii), we find that sin A has the same value in both the triangles.

Similarly, we have $\cos A = \frac{AB}{AC} = \frac{AQ}{AP}$ and $\tan A = \frac{BC}{AB} = \frac{PQ}{AQ}$

Let R be any point on AC produced. Draw RS ⊥ AB produced meeing it at S. You can verify that value of t-ratios remains the same in ΔASR also.

Thus, we conclude that the value of trigonometric ratios of an angle does not depend on the size of right triangle. They only depend on the angle.

Example 22.1: In Fig. 22.6, ΔABC is right angled at B. If AB = 5 cm, BC = 12 cm and AC = 13 cm, find the value of tan C, cosec C and sec C.

Solution: We know that

$$\tan C = \frac{\text{side opposite to } \angle C}{\text{side adjacent to } \angle C} = \frac{AB}{BC} = \frac{5}{12}$$

$$\text{cosec } C = \frac{\text{Hypotenuse}}{\text{side opposite to } \angle C} = \frac{AC}{AB} = \frac{13}{5}$$

and $$\sec C = \frac{\text{Hypotenuse}}{\text{side adjacent to } \angle C} = \frac{AC}{BC} = \frac{13}{12}$$

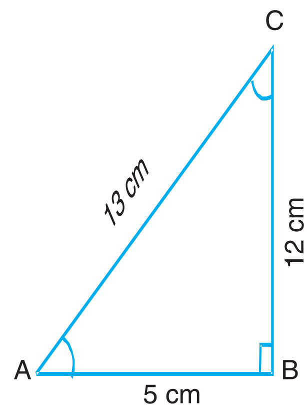


Fig. 22.6

Example 22.2 : Find the value of sin θ, cot θ and Sec θ from Fig. 22.7.

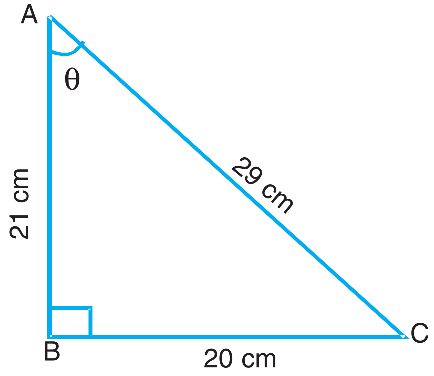


Fig. 22.7

Solution:

$$\sin\theta = \frac{\text{side opposite to } \angle\theta}{\text{Hypotenuse}} = \frac{BC}{AC} = \frac{20}{29}$$

$$\cot\theta = \frac{\text{side adjacent to } \angle\theta}{\text{side opposite to } \angle\theta} = \frac{AB}{BC} = \frac{21}{20}$$

and $\sec\theta = \frac{\text{Hypotenuse}}{\text{side adjacent to } \angle\theta} = \frac{AC}{AB} = \frac{29}{21}$

Example 22.3 : In Fig. 22.8, ΔABC is right-angled at B. If AB = 9cm, BC = 40cm and AC = 41cm, find the values cos C, cot C, tan A, and cosec A.

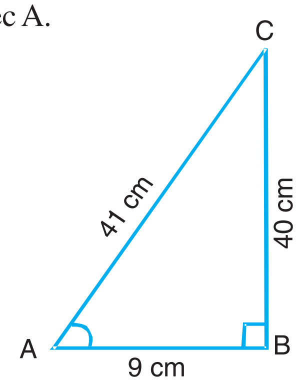


Fig. 22.8

Solution:

Now $\cos C = \frac{\text{side adjacent to } \angle C}{\text{Hypotenuse}} = \frac{BC}{AC} = \frac{40}{41}$

and $\cot C = \frac{\text{side adjacent to } \angle C}{\text{side opposite to } \angle C} = \frac{BC}{AB} = \frac{40}{9}$

With reference to ∠A, side adjacent to A is AB and side opposite to A is BC.

$\therefore \tan A = \frac{\text{side opposite to } \angle A}{\text{side adjacent to } \angle A} = \frac{BC}{AB} = \frac{40}{9}$

and $\quad \text{cosec } A = \dfrac{\text{Hypotenuse}}{\text{side opposite to } \angle A} = \dfrac{AC}{BC} = \dfrac{41}{40}$

Example 22.4 : In Fig. 22.9, ΔABC is right angled at B, $\angle A = \angle C$, $AC = \sqrt{2}$ cm and AB = 1 cm. Find the values of sin C, cos C and tan C.

Solution: In ΔABC, $\angle A = \angle C$

$\therefore \quad BC = AB = 1$ cm (Given)

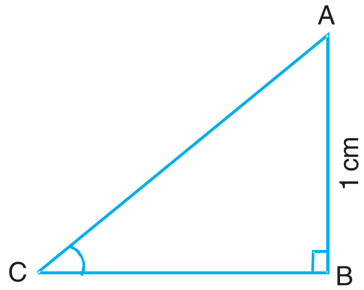


Fig. 22.9

$\therefore \quad \sin C = \dfrac{\text{side opposite to } \angle C}{\text{Hypotenuse}} = \dfrac{AB}{AC} = \dfrac{1}{\sqrt{2}}$

$\cos C = \dfrac{\text{side adjacent to } \angle C}{\text{Hypotenuse}} = \dfrac{BC}{AC} = \dfrac{1}{\sqrt{2}}$

and $\quad \tan C = \dfrac{\text{side opposite to } \angle C}{\text{side adjacent to } \angle C} = \dfrac{AB}{BC} = \dfrac{1}{1} = 1$

Remark: In the above example, we have $\angle A = \angle C$ and $\angle B = 90°$

$\therefore \quad \angle A = \angle C = 45°$,

$\therefore$ We have $\sin 45° = \cos 45° = \dfrac{1}{\sqrt{2}}$

and $\tan 45° = 1$

Example 22.5 : In Fig. 22.10. ΔABC is right-angled at C. If AB = c, AC = b and BC = a, which of the following is true?

(i) $\tan A = \dfrac{b}{c}$

(ii) $\tan A = \dfrac{c}{b}$

(iii) $\cot A = \dfrac{b}{a}$

(iv) $\cot A = \dfrac{a}{b}$

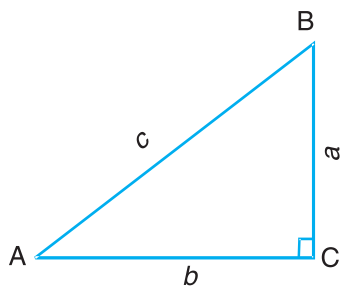


Fig. 22.10

Solution: Here $\tan A = \dfrac{\text{side opposite to } \angle A}{\text{side adjacent to } \angle A} = \dfrac{BC}{AC} = \dfrac{a}{b}$

and $\cot A = \dfrac{\text{side adjacent to } \angle A}{\text{side opposite to } \angle A} = \dfrac{b}{a}$

Hence the result (iii) i.e. $\cot A = \dfrac{b}{a}$ is true.

CHECK YOUR PROGRESS 22.1

1. In each of the following figures, ΔABC is a right triangle, right angled at B. Find all the trigonometric ratios of θ.

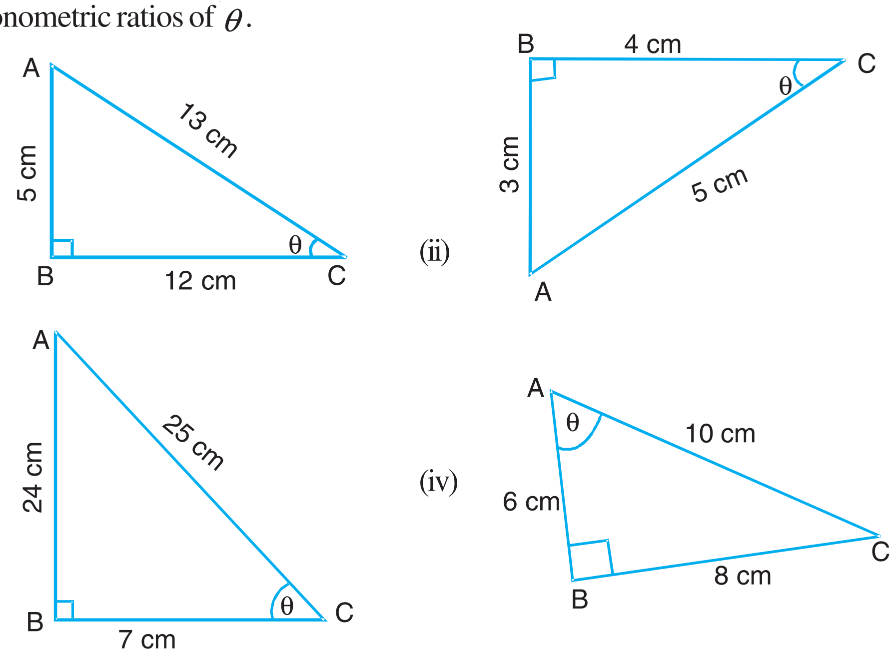


Fig. 22.11

2. In ΔABC, $\angle B = 90°$, BC = 5cm, AB = 4cm, and AC = $\sqrt{41}$ cm, find the value of sin A, cos A, and tan A.
3. In ΔABC right angled at B, if AB = 40 cm, BC = 9 cm and AC = 41 cm, find the values of sin C, cot C, cos, A and cot A.
4. In ΔABC, $\angle B = 90°$. If AB = BC = 2cm and AC = $2\sqrt{2}$ cm, find the value of sec C, cosec C, and cot C.
5. In Fig. 22.12, ΔABC is right angled at A. Which of the following is true?

   (i) $\cot C = \dfrac{13}{12}$ (ii) $\cot C = \dfrac{12}{13}$

   (iii) $\cot C = \dfrac{5}{12}$ (iv) $\cot C = \dfrac{12}{5}$

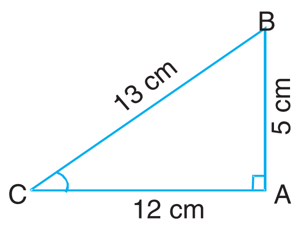


Fig. 22.12

6. In Fig. 22.13, AC = b, BC = a and AB = c. Which of the following is true?

(i) $\text{cosec A} = \frac{a}{b}$ (ii) $\text{cosec A} = \frac{c}{a}$

(iii) $\text{cosec A} = \frac{c}{b}$ (iv) $\text{cosec A} = \frac{b}{a}$.

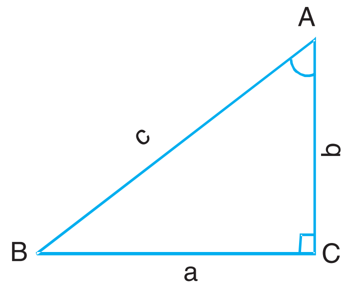


Fig. 22.13

22.2 GIVEN TWO SIDES OF A RIGHT-TRIANGLE, TO FIND TRIGONOMETRIC RATIO

When two sides of a right-triangle are given, its third side can be found out by using the Pythagoras theorem. Then we can find the trigonometric ratios of the given angle as learnt in the last section.

We take some examples to illustrate.

Example 22.6: In Fig. 22.14, ΔPQR is a right triangle, right angled at Q. If PQ = 5 cm and QR = 12 cm, find the values of sin R, cos R and tan R.

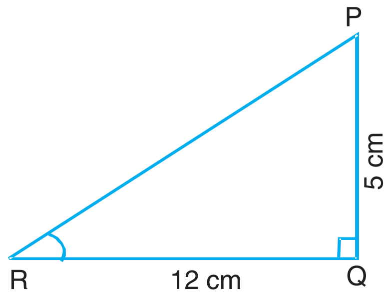


Fig. 22.14

Solution: We shall find the third side by using Pythagoras Theorem.

$\because$ ΔPQR is a right angled triangle at Q.

$$\therefore \text{PR} = \sqrt{\text{PQ}^2 + \text{QR}^2} \quad \text{(Pythagoras Theorem)}$$

$$= \sqrt{5^2 + 12^2} \text{ cm}$$

$$= \sqrt{25 + 144} \text{ cm}$$

$$= \sqrt{169} \text{ or } 13 \text{ cm}$$

We now use definition to evaluate trigonometric ratios:

$$\sin R = \frac{\text{side opposite to } \angle R}{\text{Hypotenuse}} = \frac{\text{PQ}}{\text{PR}} = \frac{5}{13}$$

$$\cos R = \frac{\text{side adjacent to } \angle R}{\text{Hypotenuse}} = \frac{\text{QR}}{\text{PR}} = \frac{12}{13}$$

and $$\tan R = \frac{\text{side opposite to } \angle R}{\text{side adjacent to } \angle R} = \frac{5}{12}$$

From the above example, we have the following:

Steps to find Trigonometric ratios when two sides of a right triangle are given.

Step1: Use Pythagoras Theorem to find the unknown (third) side of the triangle.

Step 2: Use definition of t-ratios and substitute the values of the sides.

Example 22.7 : In Fig. 22.15, ΔPQR is right-angled at Q, PR = 25cm, PQ = 7cm and $\angle PRQ = \theta$. Find the value of tan θ, cosec θ and sec θ.

Solution :

$\because$ ΔPQR is right-angled at Q

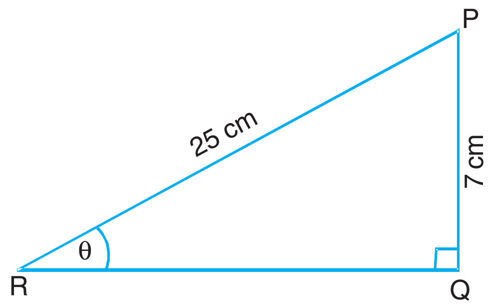


Fig. 22.15

$$\therefore\ QR = \sqrt{PR^2 - PQ^2}$$

$$= \sqrt{25^2 - 7^2}\ \text{cm}$$

$$= \sqrt{625 - 49}\ \text{cm}$$

$$= \sqrt{576}\ \text{cm}$$

$$= 24\ \text{cm}$$

$$\therefore \quad \tan\theta = \frac{PQ}{QR} = \frac{7}{24}$$

$$\text{cosec}\,\theta = \frac{PR}{PQ} = \frac{25}{7}$$

and $$\sec\theta = \frac{PR}{QR} = \frac{25}{24}$$

Example 22.8 : In ΔABC, $\angle B = 90°$. If AB = 4 cm and BC = 3 cm, find the values of sin C, cos C, cot C, tan A, sec A and cosec A. Comment on the values of tan A and cot C. Also find the value of tan A – cot C.

Solution: By Pythagoras Theorem, in ΔABC,

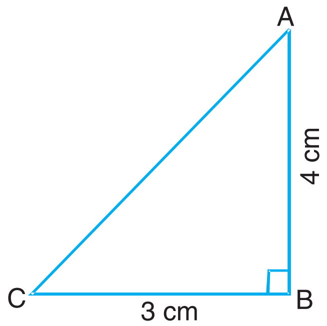


Fig. 22.16

$$AC = \sqrt{AB^2 + BC^2}$$

$$= \sqrt{4^2 + 3^2}\ \text{cm}$$

$$= \sqrt{25}\ \text{cm}$$

$$= 5\ \text{cm}$$

Now $$\sin C = \frac{AB}{AC} = \frac{4}{5}$$

$$\cos C = \frac{BC}{AC} = \frac{3}{5}$$

$$\cot C = \frac{BC}{AB} = \frac{3}{4}$$

$$\tan A = \frac{BC}{AB} = \frac{3}{4}$$

$$\sec A = \frac{AC}{AB} = \frac{5}{4}$$

and $\quad \operatorname{cosec} A = \frac{AC}{BC} = \frac{5}{3}$

The value of tan A and cot C are equal

$\therefore \quad \tan A - \cot C = 0.$

Example 22.9: In Fig. 22.17, PQR is right triangle at R. If PQ = 13cm and QR = 5cm, which of the following is true?

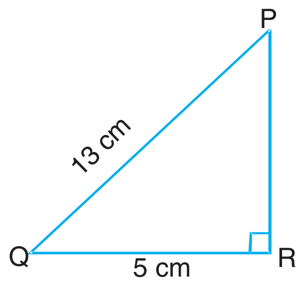


Fig. 22.17

(i) $\sin Q + \cos Q = \frac{17}{13}$ (ii) $\sin Q - \cos Q = \frac{17}{13}$

(iii) $\sin Q + \sec Q = \frac{17}{13}$ (iv) $\tan Q + \cot Q = \frac{17}{13}$

Solution: Here $PR = \sqrt{PQ^2 - QR^2} = \sqrt{13^2 - 5^2} = \sqrt{144} = 12$ cm

$\therefore \quad \sin Q = \frac{PR}{PQ} = \frac{12}{13}$ and $\cos Q = \frac{QR}{PQ} = \frac{5}{13}$

$\therefore \quad \sin Q + \cos Q = \frac{12}{13} + \frac{5}{13} = \frac{17}{13}$

Hence statement (i) i.e. $\sin Q + \cos Q = \frac{17}{13}$ is true.

CHECK YOUR PROGRESS 22.2

1. In right ΔABC, right angled at B, AC = 10 cm, and AB = 6 cm. Find the values of sin C, cos C, and tan C.

2. In ΔABC, $\angle C = 90°$, BC = 24 cm and AC = 7 cm. Find the values of sin A, cosec A and cot A.

3. In ΔPQR, $\angle Q = 90°$, PR = $10\sqrt{2}$ cm and QR = 10cm. Find the values of sec P, cot P and cosec P.

4. In ΔPQR, $\angle Q = 90°$, PQ = $\sqrt{3}$ cm and QR = 1 cm. Find the values of tan R, cosec R, sin P and sec P.

5. In ΔABC, $\angle B = 90°$, AC = 25 cm, AB = 7 cm and $\angle ACB = \theta$. Find the values of cot θ, sin θ, sec θ and tan θ.

6. In right ΔPQR, right-angled at Q, PQ = 5 cm and PR = 7 cm. Find the values of sin P, cos P, sin R and cos R. Find the value of sin P – cos R.

7. ΔDEF is a right triangle at E in Fig. 22.18. If DE = 5 cm and EF = 12 cm, which of the following is true?

(i) $\sin F = \frac{5}{12}$

(ii) $\sin F = \frac{12}{5}$

(iii) $\sin F = \frac{5}{13}$

(iv) $\sin F = \frac{12}{13}$

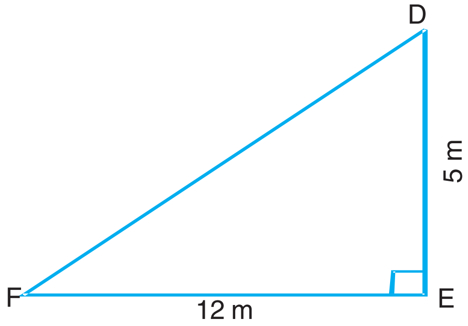


Fig. 22.18

22.3 GIVEN ONE TRIGONOMETRIC RATIO, TO FIND THE OTHERS

Sometimes we know one trigonometric ratio and we have to find the vaues of other t-ratios. This can be easily done by using the definition of t-ratios and the Pythagoras Theorem. Let us take $\sin\theta = \frac{12}{13}$. We now find the other t-ratios.

We draw a right-triangle ABC

Now $\sin\theta = \frac{12}{13}$ implies that sides AB and AC are in the ratio 12 : 13.

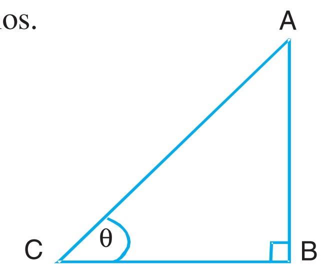


Fig. 22.19

Thus we suppose AB = 12 k and AC = 13 k.

$\therefore$ By Pythagoras Theorem,

$$BC = \sqrt{AC^2 - AB^2}$$

$$= \sqrt{(13k)^2 - (12k)^2}$$

$$= \sqrt{169k^2 - 144k^2}$$

$$= \sqrt{25k^2} = 5\text{ k}$$

Now we can find all othe t-ratios.

$$\cos\theta = \frac{BC}{AC} = \frac{5k}{13k} = \frac{5}{13}$$

$$\tan\theta = \frac{AB}{BC} = \frac{12k}{5k} = \frac{12}{5}$$

$$\text{cosec}\,\theta = \frac{AC}{AB} = \frac{13k}{12k} = \frac{13}{12}$$

$$\sec\theta = \frac{AC}{BC} = \frac{13k}{5k} = \frac{13}{5}$$

and $\cot\theta = \frac{BC}{AB} = \frac{5k}{12k} = \frac{5}{12}$

The method discussed above gives the following steps for the solution.

Steps to be followed for finding the t-ratios when one t-ratio is given.

1. Draw a right triangle ΔABC.
2. Write the given t-ratio in terms of the sides and let the constant of ratio be *k*.
3. Find the two sides in terms of *k*.
4. Use Pythagoras Theorem and find the third side.
5. Now find the remaining t-ratios by using the definition.

Let us consider some examples.

Example 22.10.: If $\cos\theta = \frac{7}{25}$, find the values of $\sin\theta$ and $\tan\theta$.

Solution : Draw a right-angled ΔABC in which $\angle B = 90°$ and $\angle C = \theta$.

We know that

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{BC}{AC} = \frac{7}{25}$$

Let BC = 7 k and AC = 25 k

Then by Pythagoras Theorem,

$$AB = \sqrt{AC^2 - BC^2}$$

$$= \sqrt{(25k)^2 - (7k)^2}$$

$$= \sqrt{625k^2 - 49k^2}$$

$$= \sqrt{576k^2} \text{ or } 24\text{ k}$$

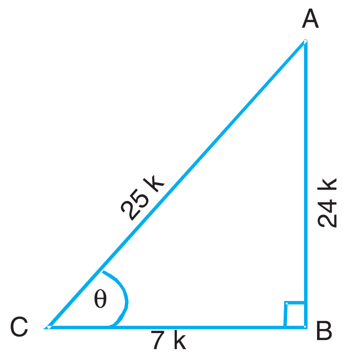


Fig. 22.20

$\therefore$ In ΔABC,

$$\sin\theta = \frac{AB}{AC} = \frac{24k}{25k} = \frac{24}{25}$$

and $$\tan\theta = \frac{AB}{BC} = \frac{24k}{7k} = \frac{24}{7}$$

Example 22.11.: If $\cot\theta = \frac{40}{9}$, find the value of $\frac{\cos\theta.\sin\theta}{\sec\theta}$.

Solution. Let ABC be a right triangle, in which $\angle B = 90°$ and $\angle C = \theta$.

We know that

$$\cot\theta = \frac{BC}{AB} = \frac{40}{9}$$

Let BC = 40k and AB = 9k

Then from right ΔABC,

$$AC = \sqrt{BC^2 + AB^2}$$

$$= \sqrt{(40k)^2 + (9k)^2}$$

$$= \sqrt{1600k^2 + 81k^2}$$

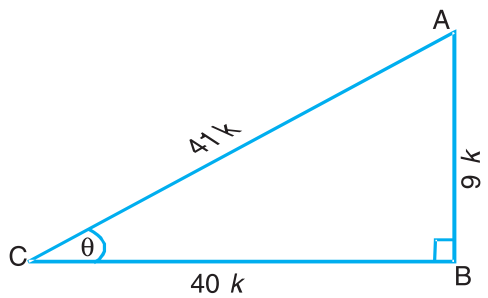


Fig. 22.21

$= \sqrt{1681k^2}$ or 41 k

Now $\quad \sin\theta = \frac{AB}{AC} = \frac{9k}{41k} = \frac{9}{41}$

$\cos\theta = \frac{BC}{AC} = \frac{40k}{41k} = \frac{40}{41}$

and $\quad \sec\theta = \frac{AC}{BC} = \frac{41k}{40k} = \frac{41}{40}$

$$\therefore \quad \frac{\cos\theta . \sin\theta}{\sec\theta} = \frac{\frac{9}{41}\times\frac{40}{41}}{\frac{41}{40}}$$

$$= \frac{9}{41}\times\frac{40}{41}\times\frac{40}{41}$$

$$= \frac{14400}{68921}$$

Example 22.12.: In PQR, $\angle Q = 90°$ and $\tan R = \frac{1}{\sqrt{3}}$. Then show that

$$\sin P \cos R + \cos P \sin R = 1$$

Solution: Let there be a right-triangle PQR, in which $\angle Q = 90°$ and $\tan R = \frac{1}{\sqrt{3}}$.

We know that

$$\tan R = \frac{PQ}{QR} = \frac{1}{\sqrt{3}}$$

Let PQ = k and QR = $\sqrt{3}$ k

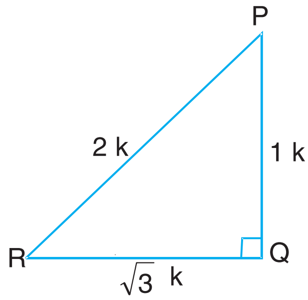


Fig. 22.22

Then, $\quad PR = \sqrt{PQ^2 + QR^2}$

$$= \sqrt{k^2 + \left(\sqrt{3}k\right)^2}$$

$$= \sqrt{k^2 + 3k^2}$$

$$= \sqrt{4k^2} \text{ or } 2\text{ k}$$

$$\therefore \quad \sin P = \frac{\text{side opposite to } \angle P}{\text{Hypotenuse}} = \frac{QR}{PR} = \frac{\sqrt{3}k}{2k} = \frac{\sqrt{3}}{2}$$

$$\cos P = \frac{\text{side adjacent to } \angle P}{\text{Hypotenuse}} = \frac{1k}{2k} = \frac{1}{2}$$

$$\sin R = \frac{\text{side opposite to } \angle R}{\text{Hypotenuse}} = \frac{PQ}{PR} = \frac{1k}{2k} = \frac{1}{2}$$

$$\text{and } \cos R = \frac{\text{side adjacent to } \angle R}{\text{Hypotenuse}} = \frac{QR}{PR} = \frac{\sqrt{3}k}{2k} = \frac{\sqrt{3}}{2}$$

$$\therefore \quad \sin P \cos R + \cos P \sin R = \frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2} + \frac{1}{2}.\frac{1}{2}$$

$$= \frac{3}{4} + \frac{1}{4} = \frac{4}{4}$$

$$= 1$$

Example 22.13.: In ΔABC, $\angle B$ is right-angle. If $AB = c$, $BC = a$ and $AC = b$, which of the following is true?

(i) $\cos C + \sin A = \frac{2b}{a}$

(ii) $\cos C + \sin A = \frac{b}{a} + \frac{a}{b}$

(iii) $\cos C + \sin A = \frac{2a}{b}$

(iv) $\cos C + \sin A = \frac{a}{b} + \frac{c}{b}$

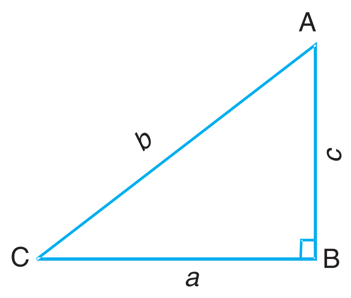


Fig. 22.23

Solution: Here $\cos C = \frac{BC}{AC} = \frac{a}{b}$

and $\sin A = \frac{BC}{AC} = \frac{a}{b}$

$\therefore \quad \cos C + \sin A = \frac{a}{b} + \frac{a}{b} = \frac{2a}{b}$

$\therefore$ Statement (iii), i.e., $\cos C + \sin A = \frac{2a}{b}$ is true.

CHECK YOUR PROGRESS 22.3

1. If $\sin\theta = \frac{20}{29}$, find the values of $\cos\theta$ and $\tan\theta$.

2. If $\tan\theta = \frac{24}{7}$, find the values of $\sin\theta$ and $\cos\theta$.

3. If $\cos A = \frac{7}{25}$, find the values of sin A and tan A.

4. If $\cos\theta = \frac{m}{n}$, find the values of $\cot\theta$ and $\text{cosec}\,\theta$.

5. If $\cos\theta = \frac{4}{5}$, evaluate $\frac{\cos\theta \,.\, \cot\theta}{1-\sec^2\theta}$.

6. If $\text{cosec}\,\theta = \frac{2}{\sqrt{3}}$, find the value of $\sin^2\theta\cos\theta + \tan^2\theta$.

7. If $\cot B = \frac{5}{4}$, then show that $\text{cosec}^2 B = 1 + \cot^2 B$.

8. ΔABC is a right triangle with $\angle C = 90^o$. If $\tan A = \frac{3}{2}$, find the values of sin B and tan B.

9. If $\tan A = \frac{1}{\sqrt{3}}$ and $\tan B = \sqrt{3}$, then show that $\cos A \cos B - \sin A \sin B = 0$.

10. If $\cot A = \frac{12}{5}$, show that $\tan^2 A - \sin^2 A = \sin^4 A \sec^2 A$.

    [Hint: Find the vlaues of tan A, sin A and sec A and substitute]

11. In Fig. 22.24, ΔABC is right-angled at vertex B. If AB = *c*, BC = *a* and CA = *b*, which of the following is true?

    (i) $\sin A + \cos A = \frac{b+c}{a}$

    (ii) $\sin A + \cos A = \frac{a+c}{b}$

    (iii) $\sin A + \cos A = \frac{a+b}{c}$

    (iv) $\sin A + \cos A = \frac{a+b+c}{b}$

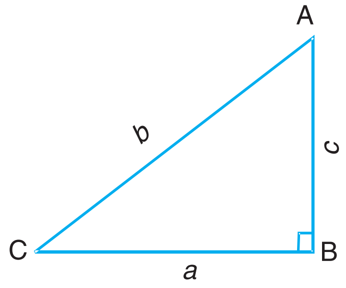


Fig. 22.24

22.4 RELATIONSHIPS BETWEEN TRIGONOMETRIC RATIOS

In a right triangle ABC, right angled at B, we have

$$\sin\theta = \frac{AB}{AC}$$

$$\cos\theta = \frac{BC}{AC}$$

and $$\tan\theta = \frac{AB}{BC}$$

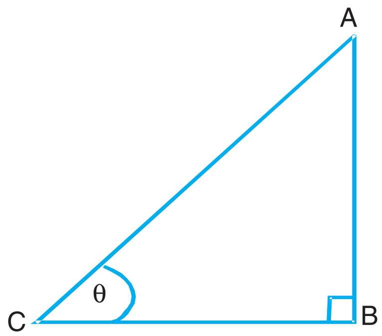


Fig. 22.25

Rewriting, $\tan\theta = \frac{AB}{BC} = \frac{AB}{AC} \div \frac{BC}{AC}$

$$= \frac{\frac{AB}{AC}}{\frac{BC}{AC}} = \frac{\sin\theta}{\cos\theta}$$

Thus, we see that $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

We can verify this result by taking AB = 3 cm, BC = 4 cm and therefore $AC = \sqrt{AB^2 + BC^2} = \sqrt{3^2 + 5^2}$ or 5 cm

$$\therefore \ \sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5} \text{ and } \tan\theta = \frac{3}{4}$$

$$\text{Now } \frac{\sin\theta}{\cos\theta} = \frac{\frac{3}{5}}{\frac{4}{5}} = \frac{3}{4} = \tan\theta.$$

Thus, the result is verified.

Again $\sin\theta = \dfrac{AB}{AC}$ gives us

$$\frac{1}{\sin\theta} = \frac{1}{\frac{AB}{AC}} = \frac{AC}{AB} = \text{cosec}\,\theta$$

Thus $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$ or $\text{cosec}\,\theta \,.\, \sin\theta = 1$

We say $\text{cosec}\,\theta$ is the reciprocal of $\sin\theta$.

Again, $\cos\theta = \dfrac{BC}{AC}$ gives us

$$\frac{1}{\cos\theta} = \frac{1}{\frac{BC}{AC}} = \frac{AC}{BC} = \sec\theta$$

Thus $\sec\theta = \dfrac{1}{\cos\theta}$ or $\sec\theta \,.\, \cos\theta = 1$

We say that $\sec\theta$ is reciprocal of $\cos\theta$.

Finally, $\tan\theta = \dfrac{AB}{BC}$ gives us

$$\frac{1}{\tan\theta} = \frac{1}{\frac{AB}{BC}} = \frac{BC}{AB} = \cot\theta$$

Thus, $\cot\theta = \frac{1}{\tan\theta}$ or $\tan\theta \,.\, \cot\theta = 1$

Also $\cot\theta = \frac{1}{\sin\theta/\cos\theta} = \frac{\cos\theta}{\sin\theta}$

We say that $\cot\theta$ is reciprocal of $\tan\theta$.

Thus, we have $\text{cosec}\,\theta$, $\sec\theta$ and $\cot\theta$ are reciprocal of $\sin\theta$, $\cos\theta$ and $\tan\theta$ respectively.

We have, therefore, established the following results:

(i) $\tan\theta = \frac{\sin\theta}{\cos\theta}$

(ii) $\text{cosec}\,\theta = \frac{1}{\sin\theta}$

(iii) $\sec\theta = \frac{1}{\cos\theta}$

(iv) $\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$

Now we can make use of the above results in finding the values of different trigonometric ratios.

Example 22.14: If $\cos\theta = \frac{1}{2}$ and $\sin\theta = \frac{\sqrt{3}}{2}$, find the values of $\text{cosec}\,\theta$, $\sec\theta$ and $\tan\theta$.

Solution: We know that

$$\text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\frac{1}{2}} = 2$$

and $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\sqrt{3}}{2}\times\frac{2}{1} = \sqrt{3}$$

Example 22.15: For a right angled triangle ABC, right angled at C, tan A = 1. Find the value of cos B.

Solution: Let us construct a right angled ΔABC in which $\angle C = 90^o$.

We have tan A = 1 (Given)

We know that

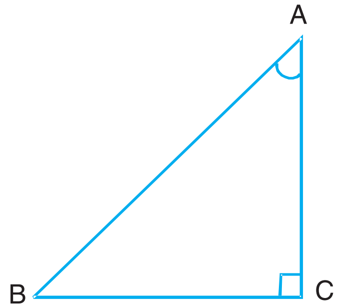


Fig. 22.26

$$\tan A = \frac{BC}{AC} = 1$$

$\therefore$ BC and AC are equal.

Let BC = AC = k

Then $AB = \sqrt{BC^2 + AC^2}$

$= \sqrt{k^2 + k^2}$

$= \sqrt{2}k$

Now $\cos B = \frac{BC}{AB} = \frac{k}{\sqrt{2}k}$

$= \frac{1}{\sqrt{2}}$

Hence $\cos B = \frac{1}{\sqrt{2}}$

CHECK YOUR PROGRESS 22.4

1. If $\sin\theta = \frac{1}{2}$ and $\cos\theta = \frac{\sqrt{3}}{2}$, find the values of $\cot\theta$ and $\sec\theta$.

2. If $\sin\theta = \frac{\sqrt{3}}{2}$ and $\tan\theta = \sqrt{3}$, find the value of $\cos^2\theta + \sin\theta\cot\theta$.

3. In a right angled ΔABC, right angled at C, $\cos A = \frac{\sqrt{3}}{2}$. Find the value of $\sin A \sin B + \cos A \cos B$.
4. If $\operatorname{cosec} A = 2$, find the value of sin A and tan A.
5. In a right angled ΔABC, right angled at B, $\tan A = \sqrt{3}$, find the value of $\tan^2 B \sec^2 A - (\tan^2 A + \cot^2 B)$

22.5 IDENTITY

We have studied about equations in algebra in our earlier classes. Recall that when two expressions are connected by '=' (equal to) sign, we get an equation. In this section, we now introduce the concept of an identity. We get an identity when two expressions are connected by the equality sign. When we say that two expressions when connected by '=' give rise to an equation as well as identity, then what is the difference between the two.

The major difference between the two is that an equation involving a variable is true for some values only whereas the equation involving a variable is true for all values of the variable, is called an identity.

Thus $x^2 - 2x + 1 = 0$ is an equation as it is true for $x = 1$.

$x^2 - 5x + 6 = 0$ is an equation as it is true for $x = 2$ and $x = 3$.

If we consider $x^2 - 5x + 6 = (x - 2)(x - 3)$, it becomes an identity as it is true for $x = 2$, $x = 3$ and say $x = 0$, $x = 10$ etc. i.e. it is true for all values of x. In the next section, we shall consider some identities in trigonometry.

22.6 TRIGONOMETRIC IDENTITIES

We know that an angle is defined with the help of the rotation of a ray from initial to final position. You have learnt to define all trigonometric ratios of an angle. Let us recall them here.

Let XOX′ and YOY′ be the rectangular axes. Let A be any point on OX. Let the ray OA start rotating in the plane in an anti-clockwise direction about the point O till it reaches the final position OA′ after some interval of time. Let $\angle A'OA = \theta$. Take any point P on the ray OA′. Draw PM ⊥ OX.

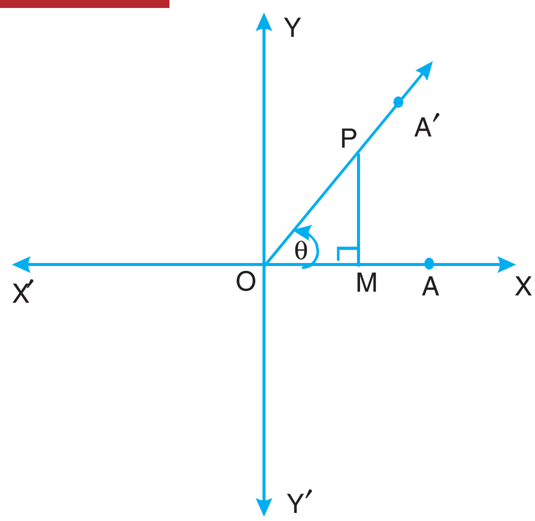


Fig. 22.27

In right angled ΔPMO,

$$\sin\theta = \frac{PM}{OP}$$

and $$\cos\theta = \frac{OM}{OP}$$

Squaring and adding, we get

$$\sin^2\theta + \cos^2\theta = \left(\frac{PM}{OP}\right)^2 + \left(\frac{OM}{OP}\right)^2$$

$$= \frac{PM^2 + OM^2}{OP^2} = \frac{OP^2}{OP^2}$$

$$= 1$$

Hence, $\sin^2\theta + \cos^2\theta = 1$ ...(1)

Also we know that

$$\sec\theta = \frac{OP}{OM}$$

and $$\tan\theta = \frac{PM}{OM}$$

Squaring and subtracting, we get

$$\sec^2\theta - \tan^2\theta = \left(\frac{OP}{OM}\right)^2 - \left(\frac{PM}{OM}\right)^2$$

$$= \frac{OP^2 - PM^2}{OM^2}$$

$$= \frac{OM^2}{OM^2}$$ [By Pythagoras Theorm, $OP^2 - PM^2 = OM^2$]

$$= 1$$

Hence, $\sec^2\theta - \tan^2\theta = 1$ ...(2)

Again, $\operatorname{cosec}\theta = \dfrac{OP}{PM}$

and $$\cot\theta = \frac{OM}{PM}$$

Squaring and subtracting, we get

$$\operatorname{cosec}^2 \theta - \cot^2 \theta = \left(\frac{OP}{PM}\right)^2 - \left(\frac{OM}{PM}\right)^2$$

$$= \frac{OP^2 - OM^2}{PM^2} = \frac{PM^2}{PM^2}$$

[By Pythagoras Theorm, $OP^2 - OM^2 = PM^2$]

$$= 1$$

Hence, $\operatorname{cosec}^2 \theta - \cot^2 \theta = 1$ ...(3)

Note: By using algebraic operations, we can write identities (1), (2) and (3) as

$\sin^2 \theta = 1 - \cos^2 \theta$ or $\cos^2 \theta = 1 - \sin^2 \theta$

$\sec^2 \theta = 1 + \tan^2 \theta$ or $\tan^2 \theta = \sec^2 \theta - 1$

and $\operatorname{cosec}^2 \theta = 1 + \cot^2 \theta$ or $\cot^2 \theta = \operatorname{cosec}^2 \theta - 1$

respectively.

We shall solve a few examples, using the above identities.

Example 22.16: Prove that

$$\tan \theta + \cot \theta = \frac{1}{\sin \theta \cos \theta}$$

Solution: L.H.S. = $\tan \theta + \cot \theta$

$$= \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta}$$

$$= \frac{\sin^2\theta + \cos^2\theta}{\sin \theta \cos \theta} = \frac{1}{\sin \theta \cos \theta} \qquad (\because \sin^2 \theta + \cos^2 \theta = 1)$$

= R.H.S.

Hence, $\tan \theta + \cot \theta = \dfrac{1}{\sin \theta \cos \theta}$

Exampe 22.17: Prove that

$$\frac{\sin A}{1 + \cos A} + \frac{1 + \cos A}{\sin A} = 2 \operatorname{cosec} A$$

Solution: $\text{L.H.S} = \dfrac{\sin A}{1 + \cos A} + \dfrac{1 + \cos A}{\sin A}$

$$= \frac{\sin^2 A + (1+\cos A)^2}{\sin A\,(1+\cos A)}$$

$$= \frac{\sin^2 A + 1 + \cos^2 A + 2\cos A}{\sin A\,(1+\cos A)}$$

$$= \frac{(\sin^2 A + \cos^2 A) + 1 + 2\cos A}{\sin A\,(1+\cos A)}$$

$$= \frac{1+1+2\cos A}{\sin A\,(1+\cos A)}$$

$$= \frac{2+2\cos A}{\sin A\,(1+\cos A)}$$

$$= \frac{2\,(1+\cos A)}{\sin A\,(1+\cos A)}$$

$$= \frac{2}{\sin A}$$

$$= 2\,\text{cosec}\,A$$

$$= \text{R.H.S.}$$

Hence, $\dfrac{\sin A}{1+\cos A} + \dfrac{1+\cos A}{\sin A} = 2\,\text{cosec}\,A$

Example 22.18: Prove that:

$$\frac{1-\sin A}{1+\sin A} = (\sec A - \tan A)^2$$

Solution: R.H.S. $= (\sec A - \tan A)^2$

$$= \left(\frac{1}{\cos A} - \frac{\sin A}{\cos A}\right)^2$$

$$= \left(\frac{1-\sin A}{\cos A}\right)^2$$

$$= \frac{(1-\sin A)^2}{\cos^2 A}$$

$$= \frac{(1-\sin A)^2}{1-\sin^2 A} \qquad (\because \cos^2 A = 1-\sin^2 A)$$

$$= \frac{(1-\sin A)^2}{(1-\sin A)(1+\sin A)}$$

$$= \frac{1-\sin A}{1+\sin A}$$

$$= \text{L.H.S.}$$

Hence, $\frac{1-\sin A}{1+\sin A} = (\sec A - \tan A)^2$

Alternative method

We can prove the identity by starting from L.H.S. in the following way:

$$\text{L.H.S.} = \frac{1-\sin A}{1+\sin A}$$

$$= \frac{1-\sin A}{1+\sin A} \times \frac{1-\sin A}{1-\sin A}$$

$$= \frac{(1-\sin A)^2}{1-\sin^2 A}$$

$$= \frac{(1-\sin A)^2}{\cos^2 A}$$

$$= \left(\frac{1-\sin A}{\cos A}\right)^2$$

$$= \left(\frac{1}{\cos A} - \frac{\sin A}{\cos A}\right)^2$$

$$= (\sec A - \tan A)^2$$

$$= \text{R.H.S.}$$

Remark: From the above examples, we get the following method for solving questions on Trigonometric identities.

Method to solve questions on Trigonometric identities

Step 1: Choose L.H.S. or R.H.S., whichever looks to be easy to simplify.

Step 2: Use different identities to simplify the L.H.S. (or R.H.S.) and arrive at the result on the other hand side.

Step 3: If you don't get the result on R.H.S. (or L.H.S.) arrive at an appropriate result and then simplify the other side to get the result already obtained.

Step 4: As both sides of the identity have been proved to be equal the identity is established.

We shall now, solve some more questions on Trigonometric identities.

Example 22.19: Prove that:

$$\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \frac{\cos\theta}{1+\sin\theta}$$

Solution: L.H.S. $= \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}$

$$= \frac{\sqrt{1-\sin\theta}}{\sqrt{1+\sin\theta}} \times \frac{\sqrt{1+\sin\theta}}{\sqrt{1+\sin\theta}}$$

$$= \frac{\sqrt{1-\sin^2\theta}}{(1+\sin\theta)}$$

$$= \frac{\sqrt{\cos^2\theta}}{1+\sin\theta} \qquad (\because 1-\sin^2\theta = \cos^2\theta)$$

$$= \frac{\cos\theta}{1+\sin\theta} = \text{R.H.S.}$$

Hence, $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}} = \frac{\cos\theta}{1+\sin\theta}$

Example 22.20: Prove that

$$\cos^4 A - \sin^4 A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$$

Solution: L.H.S. $= \cos^4 A - \sin^4 A$

$= (\cos^2 A)^2 - (\sin^2 A)^2$

$= (\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)$

$= \cos^2 A - \sin^2 A \qquad (\because \cos^2 A + \sin^2 A = 1)$

$= \text{R.H.S.}$

Again $\cos^2 A - \sin^2 A = (1 - \sin^2 A) - \sin^2 A \quad (\because \cos^2 A = 1 - \sin^2 A)$

$= 1 - 2\sin^2 A$

$= \text{R. H. S.}$

Hence $\cos^4 A - \sin^4 A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A$

Example 22.21: Prove that

$$\sec A\,(1 - \sin A)\,(\sec A + \tan A) = 1$$

Solution: L.H.S. $= \sec A\,(1 - \sin A)\,(\sec A + \tan A)$

$$= \frac{1}{\cos A}(1 - \sin A)\left(\frac{1}{\cos A} + \frac{\sin A}{\cos A}\right)$$

$$= \frac{(1 - \sin A)(1 + \sin A)}{\cos^2 A}$$

$$= \frac{1 - \sin^2 A}{\cos^2 A}$$

$$= \frac{\cos^2 A}{\cos^2 A}$$

$= 1 \ = \text{R.H.S.}$

Hence, $\sec A\,(1 - \sin A)\,(\sec A + \tan A) = 1$

Example 22.22: Prove that

$$\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \frac{1 + \sin\theta}{\cos\theta}$$

$$= \frac{\cos\theta}{1 - \sin\theta}$$

Solution: L.H.S. $= \dfrac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1}$

$$= \frac{(\tan\theta + \sec\theta) - (\sec^2\theta - \tan^2\theta)}{\tan\theta - \sec\theta + 1} \qquad (\because 1 = \sec^2\theta - \tan^2\theta)$$

$$= \frac{(\tan\theta + \sec\theta) - (\sec\theta + \tan\theta)(\sec\theta - \tan\theta)}{\tan\theta - \sec\theta + 1}$$

$$= \frac{(\tan\theta + \sec\theta)[1 - (\sec\theta - \tan\theta)]}{\tan\theta - \sec\theta + 1}$$

$$= \frac{(\tan\theta + \sec\theta)(1 - \sec\theta + \tan\theta)}{\tan\theta - \sec\theta + 1}$$

$= \tan\theta + \sec\theta$

$$= \frac{1 + \sin\theta}{\cos\theta}$$

= R.H.S.

Again $\frac{1 + \sin\theta}{\cos\theta} = \frac{(1 + \sin\theta)(1 - \sin\theta)}{\cos\theta(1 - \sin\theta)}$

$$= \frac{1 - \sin^2\theta}{\cos\theta(1 - \sin\theta)}$$

$$= \frac{\cos^2\theta}{\cos\theta(1 - \sin\theta)}$$

$$= \frac{\cos\theta}{1 - \sin\theta}$$

= R.H.S.

Hence, $\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \frac{1 + \sin\theta}{\cos\theta}$

$$= \frac{\cos\theta}{1 - \sin\theta}$$

Example 22.23: If $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$, then show that $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$.

Solution: We are given $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$

or $\cos\theta = \sqrt{2}\sin\theta + \sin\theta$

or $\cos\theta = (\sqrt{2} + 1)\sin\theta$

or $\quad \dfrac{\cos\theta}{\sqrt{2}+1} = \sin\theta$

or $\quad \sin\theta = \dfrac{\cos\theta}{\sqrt{2}+1} \times \dfrac{(\sqrt{2}-1)}{(\sqrt{2}-1)}$

or $\quad \sin\theta = \dfrac{\sqrt{2}\cos\theta - \cos\theta}{2-1}$

or $\quad \sin\theta + \cos\theta = \sqrt{2}\cos\theta$

Hence, $\cos\theta + \sin\theta = \sqrt{2}\cos\theta$.

Example 22.24: If $\tan^4\theta + \tan^2\theta = 1$, then show that

$\cos^4\theta + \cos^2\theta = 1$

Solution: We have $\quad \tan^4\theta + \tan^2\theta = 1$

or $\quad \tan^2\theta\,(\tan^2\theta + 1) = 1$

or $\quad 1 + \tan^2\theta = \dfrac{1}{\tan^2\theta} = \cot^2\theta$

or $\quad \sec^2\theta = \cot^2\theta \qquad (1 + \tan^2\theta = \sec^2\theta)$

or $\quad \dfrac{1}{\cos^2\theta} = \dfrac{\cos^2\theta}{\sin^2\theta}$

or $\quad \sin^2\theta = \cos^4\theta$

or $\quad 1 - \cos^2\theta = \cos^4\theta \qquad (\sin^2\theta = 1 - \cos^2\theta)$

or $\quad \cos^4\theta + \cos^2\theta = 1$

CHECK YOUR PROGRESS 22.5

Prove each of the following identities:

1. $(\text{cosec}^2\theta - 1)\sin^2\theta = \cos^2\theta$
2. $\sin^4 A + \sin^2 A\cos^2 A = \sin^2 A$
3. $\cos^2\theta\,(1 + \tan^2\theta) = 1$
4. $(1 + \tan^2\theta)\sin^2\theta = \tan^2\theta$

5. $\dfrac{\sin A}{1+\cos A}+\dfrac{\sin A}{1-\cos A}=2\operatorname{cosec}A$

6. $\sqrt{\dfrac{1+\cos A}{1-\cos A}}=\dfrac{1+\cos A}{\sin A}$

7. $\sqrt{\dfrac{\sec A-\tan A}{\sec A+\tan A}}=\dfrac{\cos A}{1+\sin A}$

8. $(\sin A-\cos A)^2+2\sin A\cos A=1$

9. $\cos^4\theta+\sin^4\theta-2\sin^2\theta\cos^2\theta=(2\cos^2\theta-1)^2$

10. $\dfrac{\sin A-\sin B}{\cos A+\cos B}+\dfrac{\cos A-\cos B}{\sin A+\sin B}=0$

11. $(\operatorname{cosec}\theta-\sin\theta)(\sec\theta-\cos\theta)(\tan\theta+\cos\theta)=1$

12. $\sin A(1+\tan A)+\cos A(1+\cot A)=\sec A+\operatorname{cosec}A$

13. $\dfrac{1-\cos A}{1+\cos A}=(\operatorname{cosec}A-\cot A)^2$

14. $\dfrac{\tan A}{1-\cot A}+\dfrac{\cot A}{1-\tan A}=1+\sec A\operatorname{cosec}A$

15. $\dfrac{\cot A+\operatorname{cosec}A-1}{\cot A-\operatorname{cosec}A+1}=\dfrac{1+\cos A}{\sin A}$
$$=\dfrac{\sin A}{1-\cos A}$$

16. If $\sin^2\theta+\sin\theta=1$, then show that

$$\cos^2\theta+\cos^4\theta=1$$

Select the correct alternative from the four given in each of the following questions (17 - 20):

17. $(\sin A+\cos A)^2-2\sin A\cos A$ is equal to

(i) 0 (ii) 2 (iii) 1 (iv) $\sin^2A-\cos^2A$

18. $\sin^4A-\cos^4A$ is equal to:

(i) 1 (ii) $\sin^2A-\cos^2A$ (iii) 0 (iv) $\tan^2A$

19. $\sin^2 A - \sec^2 A + \cos^2 A + \tan^2 A$ is equal to

(i) 0 (ii) 1 (iii) $\sin^2 A$ (iv) $\cos^2 A$

20. (sec A – tan A) (sec A + tan A) – (cosec A – cot A) (cosec A + cot A) is equal to

(i) 2 (ii) 1 (iii) 0 (iv) $\frac{1}{2}$

22.7 TRIGONOMETRIC RATIOS FOR COMPLEMENTARY ANGLES

In geometry, we have studied about complementary and supplementary angles. Recall that two angles are complementary if their sum is 90°. If the sum of two angles A and B is 90°, then ∠A and ∠B are complementary angles and each of them is complement of the other. Thus, angles of 20° and 70° are complementary and 20° is complement of 70° and vice versa.

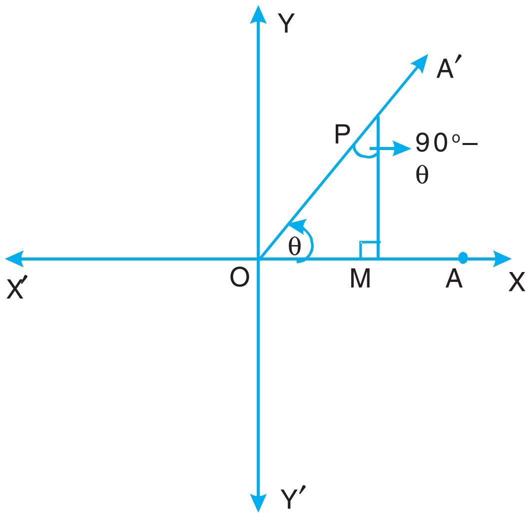


Fig. 22.28

Let XOX′ and YOY′ be a rectangular system of coordinates. Let A be any point on OX. Let ray OA be rotated in an anti clockwise direction and trace an angle θ from its initial position. Let ∠POM = θ. Draw PM ⊥ OX. Then ΔPMO is a right angled triangle.

Also, ∠POM + ∠OPM + ∠PMO = 180°

or ∠POM + ∠OPM + 90° = 180°

or ∠POM + ∠OPM = 90°

∴ ∠OPM = 90° – ∠POM = 90° – θ

Thus ∠OPM and ∠POM are complementary angles. Now in right angled triangle PMO,

$$\sin\theta = \frac{PM}{OP},\ \cos\theta = \frac{OM}{OP} \text{ and } \tan\theta = \frac{PM}{OM}$$

$$\text{cosec}\,\theta = \frac{OP}{PM},\ \sec\theta = \frac{OP}{OM} \text{ and } \cot\theta = \frac{OM}{PM}$$

For reference angle (90°–θ), we have in right ∠d ΔOPM,

$$\sin(90^\circ - \theta) = \frac{OM}{OP} = \cos\theta$$

$$\cos(90^\circ - \theta) = \frac{PM}{OP} = \sin\theta$$

$$\tan(90^\circ - \theta) = \frac{OM}{PM} = \cot\theta$$

$$\cot(90^\circ - \theta) = \frac{PM}{OM} = \tan\theta$$

$$\text{cosec}(90^\circ - \theta) = \frac{OP}{OM} = \sec\theta$$

and $$\sec(90^\circ - \theta) = \frac{OP}{PM} = \text{cosec}\,\theta$$

The above six results are known as trigonometric ratios of complementary angles. For example,

$\sin(90^\circ - 20^\circ) = \cos 20^\circ$ i.e. $\sin 70^\circ = \cos 20^\circ$

$\tan(90^\circ - 40^\circ) = \cot 40^\circ$ i.e. $\tan 50^\circ = \cot 40^\circ$ and so on.

Let us take some examples to illustrate the use of above results.

Example 22.25: Prove that $\tan 13^\circ = \cot 77^\circ$

Solution: R.H.S. $= \cot 77^\circ$

$= \cot(90^\circ - 13^\circ)$

$= \tan 13^\circ$ ....$[\because \cot(90^\circ - \theta) = \tan\theta]$

= L.H.S.

Thus, $\tan 13^\circ = \cot 77^\circ$

Example 22.26: Evaluate $\sin^2 40^\circ - \cos^2 50^\circ$

Solution: $\cos 50^\circ = \cos(90^\circ - 40^\circ)$

$= \sin 40^\circ$ ....$[\because \cos(90^\circ - \theta) = \tan\theta]$

$\therefore$ $\sin^2 40^\circ - \cos^2 50^\circ = \sin^2 40^\circ - \sin^2 40^\circ = 0$

Example 22.27: Evaluate : $\dfrac{\cos 41^\circ}{\sin 49^\circ} + \dfrac{\sec 37^\circ}{\text{cosec}\, 53^\circ}$

Solution: $\sin 49^\circ = \sin(90^\circ - 41^\circ) = \cos 41^\circ$ ...$[\because \sin(90^\circ - \theta) = \cos\theta]$

and $\operatorname{cosec} 53^\circ = \operatorname{cosec}(90^\circ - 37^\circ) = \sec 37^\circ$ ...$[\because \operatorname{cosec}(90^\circ - \theta) = \sec\theta]$

$$\therefore \quad \frac{\cos 41^\circ}{\sin 49^\circ} + \frac{\sec 37^\circ}{\operatorname{cosec} 53^\circ} = \frac{\cos 41^\circ}{\cos 41^\circ} + \frac{\sec 37^\circ}{\sec 37^\circ}$$

$$= 1 + 1 = 2$$

Example 22.28: Show that

$3 \sin 17^\circ \sec 73^\circ + 2 \tan 20^\circ \tan 70^\circ = 5$

Solution: $3 \sin 17^\circ \sec 73^\circ + 2 \tan 20^\circ \tan 70^\circ$

$= 3 \sin 17^\circ \sec(90^\circ - 17^\circ) + 2 \tan 20^\circ \tan(90^\circ - 20^\circ)$

$= 3 \sin 17^\circ \operatorname{cosec} 17^\circ + 2 \tan 20^\circ \cot 20^\circ$

...$[\because \sec(90^\circ - \theta) = \operatorname{cosec}\theta$ and $\tan(90^\circ - \theta) = \cot\theta]$

$$= 3 \sin 17^\circ . \frac{1}{\sin 17^\circ} + 2 \tan 20^\circ . \frac{1}{\tan 20^\circ}$$

$= 3 + 2 = 5$

Example 22.29: Show that $\tan 7^\circ \tan 23^\circ \tan 67^\circ \tan 83^\circ = 1$

Solution: $\tan 67^\circ = \tan(90^\circ - 23^\circ) = \cot 23^\circ$

and $\tan 83^\circ = \tan(90^\circ - 7^\circ) = \cot 7^\circ$

Now. L.H.S. $= \tan 7^\circ \tan 23^\circ \tan 67^\circ \tan 83^\circ$

$= \tan 7^\circ \tan 23^\circ \cot 23^\circ \cot 7^\circ$

$= (\tan 7^\circ \cot 7^\circ)(\tan 23^\circ \cot 23^\circ)$

$= 1 . 1 = 1$

= R.H.S.

Hence, $\tan 7^\circ \tan 23^\circ \tan 67^\circ \tan 83^\circ = 1$

Example 22.30: If $\tan A = \cot B$, prove that $A + B = 90^\circ$.

Solution: We are given

$\tan A = \cot B$

or $\tan A = \tan(90^\circ - B)$ ... $[\because \cot\theta = \tan(90^\circ - \theta)]$

$\therefore$ $A = 90^\circ - B$

or $A + B = 90^\circ$

Example 22.31: For a ΔABC, show that $\sin\left(\frac{B+C}{2}\right)=\cos\left(\frac{A}{2}\right)$, where A, B and C are interior angles of ΔABC.

Solution: We know that sum of angles of triangle is 180°.

$\therefore \quad A + B + C = 180^\circ$

or $\quad B + C = 180^\circ - A$

or $\quad \frac{B+C}{2} = 90^\circ - \frac{A}{2}$

$\therefore \quad \sin\left(\frac{B+C}{2}\right) = \sin\left(90^\circ - \frac{A}{2}\right)$

or $\quad \sin\left(\frac{B+C}{2}\right) = \cos\left(\frac{A}{2}\right)$

Example 22.32: Prove that $\frac{\cos\theta}{\sin(90^\circ-\theta)} + \frac{\sin\theta}{\cos(90^\circ-\theta)} = 2$.

Solution: L.H.S. $= \frac{\cos\theta}{\sin(90^\circ-\theta)} + \frac{\sin\theta}{\cos(90^\circ-\theta)}$

$= \frac{\cos\theta}{\cos\theta} + \frac{\sin\theta}{\sin\theta}$ ... [∵ sin (90° – θ) = cos θ and cos (90° – θ) = sin θ]

$= 1 + 1 = 2$

= R.H.S.

Hence, $\frac{\cos\theta}{\sin(90^\circ-\theta)} + \frac{\sin\theta}{\cos(90^\circ-\theta)} = 2$

Example 22.33: Show that $\frac{\sin(90^\circ-\theta)}{\text{cosec}(90^\circ-\theta)} + \frac{\cos(90^\circ-\theta)}{\sec(90^\circ-\theta)} = 1$

Solution: L.H.S. $= \frac{\sin(90^\circ-\theta)}{\text{cosec}(90^\circ-\theta)} + \frac{\cos(90^\circ-\theta)}{\sec(90^\circ-\theta)}$

$= \frac{\cos\theta}{\sec\theta} + \frac{\sin\theta}{\text{cosec}\,\theta}$ ...[∵ sin (90° – θ) = cos θ, cos (90° – θ) = sin θ,

cosec (90° – θ) = sec θ and sec (90° – θ) = cosec θ]

$$= \frac{\cos\theta}{\cos\theta} + \frac{\sin\theta}{\sin\theta} = \cos^2\theta + \sin^2\theta \ = 1$$

= R.H.S.

Hence, $\frac{\sin(90^\circ - \theta)}{\mathrm{cosec}(90^\circ - \theta)} + \frac{\cos(90^\circ - \theta)}{\sec(90^\circ - \theta)} = 1$

Example 22.34: Simplify:

$$\frac{\cos(90^\circ - \theta)\sec(90^\circ - \theta)\tan\theta}{\mathrm{cosec}(90^\circ - \theta)\sin(90^\circ - \theta)\cot(90^\circ - \theta)} + \frac{\tan(90^\circ - \theta)}{\cot\theta}$$

Solution: The given expression

$$= \frac{\cos(90^\circ - \theta)\sec(90^\circ - \theta)\tan\theta}{\mathrm{cosec}(90^\circ - \theta)\sin(90^\circ - \theta)\cot(90^\circ - \theta)} + \frac{\tan(90^\circ - \theta)}{\cot\theta}$$

$$= \frac{\sin\theta \mathrm{in}\theta.\mathrm{co}\ \theta.\tan\theta}{\sec\theta \mathrm{ec}\theta.\mathrm{cotan}\,\theta} + \frac{\cot\theta}{\cot\theta}$$ ...[$\because$ sin θ . cos θ = 1 and sec θ . cos θ = 1]

= 1 + 1

= 2

Example 22.35: Express tan 68° + sec 68° in terms of angles between 0° and 45°.

Solution: We know that

tan (90° – θ) = cot θ

and sec (90° – θ) = cosec θ

∴ tan 68° = tan (90° – 22°) = cot 22°

and sec 68° = sec (90° – 22°) = cosec 22°

Hence tan 68° + sec 68° = cot 22° + cosec 22°.

Remark: While using notion of complementary angles, usually we change that angle which is > 45° to its complement.

Example 22.36: If tan 2A = cot (A – 18°) where 2A is an acute angle, find the value of A.

Solution: We are given tan 2A = cot (A – 18°)

or cot (90° – 2A) = cot (A – 18°) ...[$\because$ cot (90° – 2A = tan 2A]

$\therefore \quad 90^\circ - 2A = A - 18^\circ$

or $\quad 3A = 90^\circ + 18^\circ$

or $\quad 3A = 108^\circ$

or $\quad A = 36^\circ$

CHECK YOUR PROGRESS 22.6

1. Show that:

   (i) $\cos 55^\circ = \sin 35^\circ$

   (ii) $\sin^2 11^\circ - \cos^2 79^\circ = 0$

   (iii) $\cos^2 51^\circ - \sin^2 39^\circ = 0$

2. Evaluate each of the following:

   (i) $\dfrac{3\sin 19^\circ}{\cos 71^\circ}$ (ii) $\dfrac{\tan 65^\circ}{2\cot 25^\circ}$ (iii) $\dfrac{\cos 89^\circ}{3\sin 1^\circ}$

   (iv) $\cos 48^\circ - \sin 42^\circ$ (v) $\dfrac{3\sin 5^\circ}{\cos 85^\circ} + \dfrac{2\tan 33^\circ}{\cot 57^\circ}$

   (vi) $\dfrac{\cot 54^\circ}{\tan 36^\circ} + \dfrac{\tan 20^\circ}{\cot 70^\circ} - 2$

   (vii) $\sec 41^\circ \sin 49^\circ + \cos 49^\circ \operatorname{cosec} 41^\circ$

   (viii) $\dfrac{\cos 75^\circ}{\sin 15^\circ} + \dfrac{\sin 12^\circ}{\cos 78^\circ} - \dfrac{\cos 18^\circ}{\sin 72^\circ}$

3. Evaluate each of the following:

   (i) $\left(\dfrac{\sin 47^\circ}{\cos 43^\circ}\right)^2 + \left(\dfrac{\cos 43^\circ}{\sin 47^\circ}\right)^2$

   (ii) $\dfrac{\cos^2 20^\circ + \cos^2 70^\circ}{3(\sin^2 59^\circ + \sin^2 31^\circ)}$

4. Prove that:

   (i) $\sin\theta \cos(90^\circ - \theta) + \cos\theta \sin(90^\circ - \theta) = 1$

(ii) $\cos\theta \cos(90^\circ - \theta) - \sin\theta \sin(90^\circ - \theta) = 0$

(iii) $\dfrac{\cos(90^\circ - \theta)}{1+\sin(90^\circ - \theta)} + \dfrac{1+\sin(90^\circ - \theta)}{\cos(90^\circ - \theta)} = 2\text{cosec}\theta$

(iv) $\sin(90^\circ - \theta)\cos(90^\circ - \theta) = \dfrac{\tan(90^\circ - \theta)}{1+\tan^2(90^\circ - \theta)}$

(v) $\tan 45^\circ \tan 13^\circ \tan 77^\circ \tan 85^\circ = 1$

(vi) $2 \tan 15^\circ \tan 25^\circ \tan 65^\circ \tan 75^\circ = 2$

(vii) $\sin 20^\circ \sin 70^\circ - \cos 20^\circ \cos 70^\circ = 0$

5. Show that $\sin(50^\circ + \theta) - \cos(40^\circ - \theta) = 0$

6. If $\sin A = \cos B$ where A and B are acute angles, prove that $A + B = 90^\circ$.

7. In a ΔABC, prove that

   (i) $\tan\left(\dfrac{B+C}{2}\right) = \cot\left(\dfrac{A}{2}\right)$

   (ii) $\cos\left(\dfrac{A+B}{2}\right) = \sin\left(\dfrac{C}{2}\right)$

8. Express $\tan 59^\circ + \text{cosec } 85^\circ$ in terms of trigonometric ratios of angles between 0° and 45°.

9. Express $\sec 46^\circ - \cos 87^\circ$ in terms of trigonometric ratios of angles between 0° and 45°.

10. Express $\sec^2 62^\circ + \sec^2 69^\circ$ in terms of trigonometric ratios of angles between 0° and 45°.

Select the correct alternative for each of the following questions (11-12):

11. The value of $\dfrac{\sin 40^\circ}{2\cos 50^\circ} - \dfrac{2\sec 41^\circ}{3\text{cosec}49^\circ}$ is

    (i) -1 (ii) $\dfrac{1}{6}$ (iii) $-\dfrac{1}{6}$ (iv) 1

12. If $\sin(\theta + 36^\circ) = \cos\theta$, where $\theta + 36^\circ$ is an acute angle, then θ is

    (i) 54° (ii) 18° (iii) 21° (iv) 27°

LET US SUM UP

- In a right angled triangle, we define trignometric ratios as under:

$$\sin\theta = \frac{\text{side opposite to angle } \theta}{\text{Hypotenuse}} = \frac{AB}{AC}$$

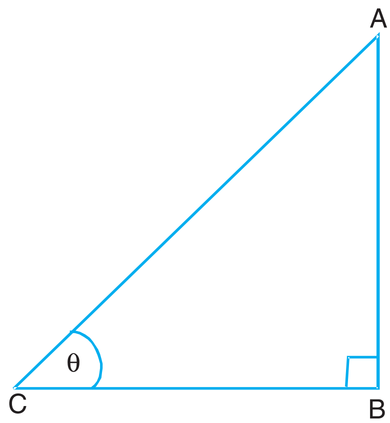
$$\cos\theta = \frac{\text{side adjacent to angle } \theta}{\text{Hypotenuse}} = \frac{BC}{AC}$$

$$\tan\theta = \frac{\text{side opposite to angle } \theta}{\text{side adjacent to angle } \theta} = \frac{AB}{BC}$$

$$\cot\theta = \frac{\text{side adjacent to angle } \theta}{\text{side opposite to angle } \theta} = \frac{BC}{AB}$$

$$\sec\theta = \frac{\text{Hypotenuse}}{\text{side adjacent to angle } \theta} = \frac{AC}{BC}$$

$$\operatorname{cosec}\theta = \frac{\text{Hypotenuse}}{\text{side opposite to angle } \theta} = \frac{AC}{AB}$$



- The following relationships exist between different trigonometric ratios:

(i) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ (ii) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

(iii) $\sec\theta = \dfrac{1}{\cos\theta}$ (iv) $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

(v) $\cot\theta = \dfrac{1}{\tan\theta}$

- The trigonometric identities are:

(i) $\sin^2\theta + \cos^2\theta = 1$

(ii) $\sec^2\theta - \tan^2\theta = 1$

(iii) $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

- Two angles, whose sum is 90°, are called complementary angles.

- $\sin(90^\circ - A) = \cos A$, $\cos(90^\circ - A) = \sin A$ and $\tan(90^\circ - A) = \cot A$.
- $\text{cosec}(90^\circ - A) = \sec A$, $\sec(90^\circ - A) = \text{cosec} A$ and $\cot(90^\circ - A) = \tan A$

Supportive website:

- http://www.wikipedia.org
- http://mathworld:wolfram.com

TERMINAL EXERCISE

1. If $\sin A = \frac{4}{5}$, find the values of cos A and tan A.

2. If $\tan A = \frac{20}{21}$, find the values of cosec A and sec A.

3. If $\cot\theta = \frac{3}{4}$, find the value of $\sin\theta + \cos\theta$.

4. If $\sec\theta = \frac{m}{n}$, find the values of $\sin\theta$ and $\tan\theta$.

5. If $\cos\theta = \frac{3}{5}$, find the value of

   $$\frac{\sin\theta\tan\theta - 1}{2\tan^2\theta}$$

6. If $\sec\theta = \frac{5}{4}$, find the value of $\frac{\tan\theta}{1+\tan\theta}$

7. If $\tan A = 1$ and $\tan B = \sqrt{3}$, find the value of $\cos A \cos B - \sin A \sin B$.

Prove each of the following identities (8 –20):

8. $(\sec\theta + \tan\theta)(1 - \sin\theta) = \cos\theta$.

9. $$\frac{\cot\theta}{1-\tan\theta} = \frac{\text{cosec}\theta}{\sec\theta}$$

10. $$\frac{1-\cos\theta}{1+\cos\theta} = (\text{cosec}\theta - \cot\theta)^2$$

11. $\dfrac{\tan\theta+\sin\theta}{\tan\theta-\sin\theta}=\dfrac{\sec\theta+1}{\sec\theta-1}$

12. $\dfrac{\tan A+\cot B}{\cot A+\tan B}=\tan A\cot B$

13. $\sqrt{\dfrac{1+\cos A}{1-\cos A}}=\operatorname{cosec}A+\cot A$

14. $\sqrt{\dfrac{\operatorname{cosec}A+1}{\operatorname{cosec}A-1}}=\dfrac{\cos A}{1-\sin A}$

15. $\sin^3A-\cos^3A=(\sin A-\cos A)\,(1+\sin A\cos A)$

16. $\dfrac{\cos A}{1-\tan A}+\dfrac{\sin A}{1-\cot A}=\cos A+\sin A$

17. $\sqrt{\dfrac{\sec A-1}{\sec A+1}}+\sqrt{\dfrac{\sec A+1}{\sec A-1}}=2\operatorname{cosec}A$

18. $(\operatorname{cosec}A-\sin A)(\sec A-\cos A)=\dfrac{1}{\tan A+\cot A}$

19. $(1+\cot\theta-\operatorname{cosec}\theta)\,(1+\tan\theta+\sec\theta)=2$

20. $2(\sin^6\theta+\cos^6\theta)-3(\sin^4\theta+\cos^4\theta)+1=0$

21. If $\sec\theta+\tan\theta=p$, show that $\sin\theta=\dfrac{p^2-1}{p^2+1}$

22. Prove that $\dfrac{\cos(90^\circ-A)}{1+\sin(90^\circ-A)}+\dfrac{1+\sin(90^\circ-A)}{\cos(90^\circ-A)}=2\sec(90^\circ-A)$

23. Prove that $\dfrac{\sin(90^\circ-A).\cos(90^\circ-A)}{\tan A}=\sin^2(90^\circ-A)$

24. If $\tan\theta=\dfrac{3}{4}$ and $\theta+\alpha=90^\circ$, find the value of $\cot\alpha$.

25. If $\cos(2\theta+54^\circ)=\sin\theta$ and $(2\theta+54^\circ)$ is an acute angle, find the value of θ.

26. If $\sec Q=\operatorname{cosec}P$ and P and Q are acute angles, show that $P+Q=90^\circ$.

ANSWERS TO CHECK YOUR PROGRESS

22.1

1. (i) $\sin\theta = \frac{5}{13}, \cos\theta = \frac{12}{13}, \tan\theta = \frac{5}{12}$

$\text{cosec}\,\theta = \frac{13}{5}, \sec\theta = \frac{13}{12}$ and $\cot\theta = \frac{12}{5}$

(ii) $\sin\theta = \frac{3}{5}, \cos\theta = \frac{4}{5}, \tan\theta = \frac{3}{4}$

$\text{cosec}\,\theta = \frac{5}{3}, \sec\theta = \frac{5}{4}$ and $\cot\theta = \frac{4}{3}$

(iii) $\sin\theta = \frac{24}{25}, \cos\theta = \frac{7}{25}, \tan\theta = \frac{24}{7}$

$\text{cosec}\,\theta = \frac{25}{24}, \sec\theta = \frac{25}{7}$ and $\cot\theta = \frac{7}{24}$

(iv) $\sin\theta = \frac{4}{5}, \cos\theta = \frac{3}{5}, \tan\theta = \frac{4}{3}$

$\text{cosec}\,\theta = \frac{5}{4}, \sec\theta = \frac{5}{3}$ and $\cot\theta = \frac{3}{4}$

2. $\sin A = \frac{5}{\sqrt{41}}, \cos A = \frac{4}{\sqrt{41}}$ and $\tan A = \frac{5}{4}$

3. $\sin C = \frac{40}{41}, \cot C = \frac{9}{40}, \cos A = \frac{40}{41}$ and $\cot A = \frac{40}{9}$

4. $\sec C = \sqrt{2}, \text{cosec}\, C = \sqrt{2}$ and $\cot C = 1$

5. (iv)

6. (ii)

22.2

1. $\sin C = \frac{3}{5}, \cos C = \frac{4}{5}$ and $\tan C = \frac{3}{4}$

2. $\sin A = \frac{24}{25}$, $\operatorname{cosec} A = \frac{25}{24}$ and $\cot A = \frac{7}{24}$

3. $\sec P = \sqrt{2}$, $\cot P = 1$, and $\operatorname{cosec} P = \sqrt{2}$

4. $\tan R = \sqrt{3}$, $\operatorname{cosec} R = \frac{2}{\sqrt{3}}$, $\sin P = \frac{1}{2}$ and $\sec P = \frac{2}{\sqrt{3}}$

5. $\cot\theta = \frac{24}{7}$, $\sin\theta = \frac{7}{25}$, $\sec\theta = \frac{25}{24}$, and $\tan\theta = \frac{7}{24}$

6. $\sin P = \frac{2\sqrt{6}}{7}$, $\cos P = \frac{5}{7}$, $\sin R = \frac{5}{7}$ and $\cos R = \frac{2\sqrt{6}}{7}$, $\sin P - \cos R = 0$

7. (iii)

22.3

1. $\cos\theta = \frac{21}{29}$ and $\tan\theta = \frac{20}{21}$

2. $\sin\theta = \frac{24}{25}$ and $\cos\theta = \frac{7}{25}$

3. $\sin A = \frac{24}{25}$ and $\tan A = \frac{24}{7}$

4. $\cot\theta = \frac{m}{\sqrt{n^2 - m^2}}$ and $\operatorname{cosec}\theta = \frac{n}{\sqrt{n^2 - m^2}}$

5. $-\frac{256}{135}$

6. $\frac{27}{8}$

7. $\sin B = \frac{2}{\sqrt{13}}$ and $\tan B = \frac{2}{3}$

11. (ii)

22.4

1. $\cot\theta = \sqrt{3}$ and $\sec\theta = \frac{2}{\sqrt{3}}$

2. $\frac{3}{4}$

3. $\frac{\sqrt{3}}{2}$

4. $\sin A = \frac{1}{2}$ and $\tan A = \frac{1}{\sqrt{3}}$

5. $-\frac{14}{3}$

22.5

17. (iii)

18. (ii)

19. (i)

20. (iii)

22.6

1. (i) 3 (ii) $\frac{1}{2}$ (iii) $\frac{1}{3}$ (iv) 0

   (v) 5 (vi) 0 (vii) 2 (viii) 1

3. (i) 2 (ii) $\frac{1}{3}$

8. $\cot 31^\circ + \sec 5^\circ$

9. $\operatorname{cosec} 44^\circ - \sin 3^\circ$

10. $\operatorname{cosec}^2 28^\circ + \operatorname{cosec}^2 21^\circ$

11. (ii)

12. (iv)

ANSWERS TO TERMINAL EXERCISE

1. $\cos A = \frac{3}{5}$ and $\tan A = \frac{4}{3}$

2. $\text{cosec } A = \frac{29}{20}$ and $\sec A = \frac{29}{21}$

3. $\frac{7}{5}$

4. $\sin\theta = \frac{\sqrt{m^2 - n^2}}{m}$ and $\tan\theta = \frac{\sqrt{m^2 - n^2}}{n}$

5. $\frac{3}{160}$

6. $\frac{3}{7}$

7. $\frac{1-\sqrt{3}}{2\sqrt{2}}$

24. $\frac{3}{4}$

25. 12^{o}